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STABILITY ANALYSIS FOR LAMINAR FLOW CONTROL

Part I

David J. Benney and Steven A. Orszag

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CAMBRIDGE HYDRODYNAMICS, INC.
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16. Abstract

This report develops the basic equations for the stability analysis of flow over three-dimensional swept wings and then surveys numerical methods for their solution. The equations for nonlinear stability analysis of three-dimensional disturbances in compressible, three-dimensional, non-parallel flows are given. Efficient and accurate numerical methods for the solution of the equations of stability theory are surveyed and analyzed.

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STABILITY ANALYSIS FOR LAMINAR FLOW CONTROL - PART I

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1. INTRODUCTION

The theory of stability of quasi-parallel flows has been subject to extensive study during recent years. The problem is nonlinear and analytical progress is difficult without the introduction of some form of approximation. For this reason, much attention has been given to the linearized problem so that the study of Orr-Sommerfeld-like equations has been the dominant theme. In particular, accurate methods for the calculation of eigenvalues and eigenfunctions has been and remains an important task. One of the two main goals of the present work is to assess and recommend efficient and accurate numerical methods for the calculation of eigenvalues and, when necessary, eigenfunctions. A full discussion of these matters is given in Sects. 5-8.

Beyond the linear regime, various nonlinear theories have been proposed and each of these have advantages and defects depending on the particular physical problem under study. Here we are concerned with boundary layer stability predictions and for this purpose the technique which will be adopted in Sects. 2-4 is the standard one in which the nonlinear evolution of the basic amplitude is studied by simple perturbation techniques. The idea of this method originated in the work of Stuart[1] and Watson[2]. When the non-parallel aspects of the flow are incorporated into this system of equations, the result is a nonlinear space-time evolution equation for the amplitude.[3-8]

In Sects. 2-4, we formulate a theory for the nonlinear, nonparallel stability of a very general class of boundary layer flows. The effects of nonlinearity, compressibility, and the quasi-parallel nature of the flow are all included. The theory given here is an important first step in trying to obtain a realistic approach to the prediction of transition in three-dimensional boundary-layer flows like those encountered on laminar-flow-control aircraft.

2. STABILITY OF QUASI-PARALLEL INCOMPRESSIBLE FLOW

It is instructive to work out the stability theory for incompressible flows before proceeding in Sects. 3-4 to the more complicated case of compressible flows. Here we formulate the nonlinear, nonparallel stability theory of two-dimensional disturbances in incompressible boundary layer flows. The basic ideas used here carry over directly to the problem of three-dimensional disturbances in a compressible flow.

It is convenient to use a streamfunction ψ, so that the equation for ψ is

$$(\nabla^2\psi)_t + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = \nu\nabla^4\psi \tag{2.1}$$

If $\bar{\psi}(y,X)$ is the basic flow with X as the slow variable (to indicate non-parallel effects), and $\psi(y,\theta,X,T)$ is the perturbation streamfunction (with T as the slow time variable and θ as the phase of the perturbation), then

$$\begin{aligned}
&(\theta_x\bar{\psi}_y + \theta_T)(\psi_{yy\theta} + \theta_x^2\,\psi_{\theta\theta\theta}) - \theta_x\bar{\psi}_{yyy}\,\psi_\theta \\
&\quad - \nu(\psi_{yyyy} + 2\theta_x^2\,\psi_{yy\theta\theta} + \theta_x^4\,\psi_{\theta\theta\theta\theta}) \\
&\quad + \mu[2\theta_T\theta_x\psi_{\theta\theta x} + \psi_{yyT} + \theta_x^2\,\psi_{\theta\theta T} + (\theta_T\theta_{xx} + 2\theta_x\theta_{xT})\psi_{\theta\theta} \\
&\qquad + \bar{\psi}_y(\psi_{yyx} + 3\theta_x^2\,\psi_{\theta\theta x} + 3\theta_x\theta_{xx}\psi_{\theta\theta}) + \bar{\psi}_{yyx}\psi_y \\
&\qquad - \bar{\psi}_x(\psi_{yyy} + \theta_x^2\,\psi_{y\theta\theta}) - \nu(4\theta_x\psi_{yy\theta x} + 2\theta_{xx}\psi_{yy\theta})' \\
&\quad + \epsilon[\theta_x\psi_y(\psi_{yy\theta} + \theta_x^2\,\psi_{\theta\theta\theta}) - \theta_x\psi_\theta(\psi_{yyy} + \theta_x^2\,\psi_{y\theta\theta})] = 0
\end{aligned} \tag{2.2}$$

Here we have assumed that the phase θ is of the form

$$\theta = \Theta(X,T)/\mu ,$$

$X = \mu x$, $\mu = \delta/L$ where δ is the boundary layer thickness and L is a measure of the distance over which non-parallel flow effects are important, and ε is a parameter specifying the amplitude of the nonlinearity. In formulating (2.2), we have also used the relations

$$\frac{\partial}{\partial x} = \mu \frac{\partial}{\partial X} + \Theta_X \frac{\partial}{\partial \theta} , \quad \frac{\partial}{\partial t} = \mu \frac{\partial}{\partial T} + \Theta_T \frac{\partial}{\partial \theta}$$

In order to solve (2.2), ψ is expanded as

$$\begin{aligned}
\psi = {} & A\psi^{(1)}e^{i\theta} + A^*\psi^{(1)*}e^{-i\theta} \\
& + \varepsilon[AA^*\psi^{(0)} + A^2\psi^{(2)}e^{2i\theta} + A^{*2}\psi^{(2)*}e^{-2i\theta}] \\
& + \varepsilon^2[A^3\psi^{(3)}e^{3i\theta} + A^{*3}\psi^{(3)*}e^{-3i\theta} + A^2A^*\psi^{(11)}e^{i\theta} \\
& \qquad + AA^{*2}\psi^{(11)*}e^{-i\theta}] \\
& + \mu[A_X\psi_1^{(1)}e^{i\theta} + A_X^*\psi_1^{(1)*}e^{-i\theta} + A\psi_2^{(1)}e^{i\theta} \\
& \qquad + A^*\psi_2^{(1)}e^{-i\theta}] + \ldots
\end{aligned} \tag{2.3}$$

where $A(X,T)$ satisfies the slow space-time scale equation

$$A_T = \alpha_1 A_X + \alpha_2 A + \lambda A^2 A^* \tag{2.4}$$

If nonlinear and nonparallel effects appear at the same order, we must take $\mu = \varepsilon^2$. Making this assumption, the rest of the calculations proceed by equating terms order by order in (2.2).

The first few boundary-value problems obtained from (2.2) are:

$$i(\theta_x\bar{\psi}_y + \theta_T)(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)}) - i\theta_x\bar{\psi}_{yyy}\psi^{(1)}$$

$$- \nu(\psi^{(1)}_{yyyy} - 2\theta_x^2\,\psi^{(1)}_{yy} + \theta_x^4\,\psi^{(1)}) = 0 \qquad (2.5)$$

$$- \nu\psi^{(0)}_{yyyy} + \theta_x[i\psi^{(1)*}_y(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)}) - i\psi^{(1)}_y(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)})$$

$$+ i\psi^{(1)*}(\psi^{(1)}_{yyy} - \theta_x^2\,\psi^{(1)}_y)$$

$$- i\psi^{(1)}(\psi^{(1)*}_{yyy} - \theta_x^2\,\psi^{(1)*}_y)] = 0 \qquad (2.6)$$

$$- \nu[\psi^{(2)}_{yyyy} - 8\theta_x^2\,\psi^{(2)}_{yy} + 16\theta_x^4\,\psi^{(2)}]$$

$$+ i\theta_x[\psi^{(1)}_y(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)}) - \psi^{(1)}(\psi^{(1)}_{yyy} - \theta_x^2\,\psi^{(1)}_y)] = 0 \qquad (2.7)$$

$$i(\theta_x\bar{\psi}_y + \theta_T)(\psi^{(11)}_{yy} - \theta_x^2\,\psi^{(11)}) - i\theta_x\bar{\psi}_{yyy}\psi^{(11)}$$

$$- \nu(\psi^{(11)}_{yyyy} - 2\theta_x^2\,\psi^{(11)}_{yy} + \theta_x^4\,\psi^{(11)}) + \lambda(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)})$$

$$+ \theta_x[\psi^{(2)}_y(-i)(\psi^{(1)*}_{yy} - \theta_x^2\,\psi^{(1)*}) + \psi^{(1)*}_y(2i)(\psi^{(2)}_{yy} - 4\theta_x^2\,\psi^{(2)})$$

$$- 2i\psi^{(2)}(\psi^{(1)*}_{yyy} - \theta_x^2\,\psi^{(1)*}_y) + i\psi^{(1)*}(\psi^{(2)}_{yyy} - 4\theta_x^2\,\psi^{(2)}_y)$$

$$+ \psi^{(0)}_y(i)(\psi^{(1)}_{yy} - \theta_x^2\,\psi^{(1)}) - i\psi^{(0)}_{yyy}\,\psi^{(1)}] = 0 \qquad (2.8)$$

$$i(\theta_x\bar{\psi}_y + \theta_T)(\psi_{1yy}^{(1)} - \theta_x^2\,\psi_1^{(1)}) - i\theta_x\bar{\psi}_{yyy}\psi_1^{(1)}$$

$$- \nu(\psi_{1yyyy}^{(1)} - 2\theta_x^2\,\psi_{1yy}^{(1)} + \theta_x^4\,\psi_1^{(1)}) + \alpha_1(\psi_{yy}^{(1)} - \theta_x^2\,\psi^{(1)})$$

$$- 2\theta_T\theta_x\psi^{(1)} + \bar{\psi}_y(\psi_{yy}^{(1)} - 3\theta_x^2\,\psi^{(1)}) - \nu 4i\theta_x\psi_{yy}^{(1)} = 0 \qquad (2.9)$$

$$i(\theta_x\bar{\psi}_y + \theta_T)(\psi_{2yy}^{(1)} - \theta_x^2\,\psi_2^{(1)}) - i\theta_x\bar{\psi}_{yy}\psi_2^{(1)}$$

$$- \nu(\psi_{2yyyy}^{(1)} - 2\theta_x^2\,\psi_{2yy}^{(1)} + \theta_x^4\,\psi_2^{(1)}) + \alpha_2(\psi_{yy}^{(1)} - \theta_x^2\,\psi^{(1)})$$

$$- 2\theta_T\theta_x\psi_x^{(1)}(\theta_T\theta_{xx} + 2\theta_x\theta_{TT})\psi^{(1)}$$

$$+ \bar{\psi}_y(\psi_{yyx}^{(1)} - 3\theta_x^2\,\psi_x^{(1)} - 3\theta_x\theta_{xx}\psi^{(1)}) + \bar{\psi}_{yyx}\psi_y^{(1)}$$

$$- \psi_x(\psi_{yyy}^{(1)} - \theta_x^2\,\psi_y^{(1)}) - i\nu(4\theta_x\psi_{yyx}^{(1)} + 2\theta_{xx}\psi_{yy}^{(1)})$$

$$+ i\theta_x\bar{\psi}_y(\psi_{yy}^{(1)} - \theta_x^2\,\psi^{(1)}) - i\theta_x\bar{\psi}_{yyy}\psi^{(1)} = 0\ . \qquad (2.10)$$

Eq. (2.5) is the homogeneous problem

$$\mathcal{L}\,\psi^{(1)} = 0 \qquad (2.11)$$

where $\mathcal{L}$ is the Orr-Sommerfeld fourth-order differential operator. The usual form of the Orr-Sommerfeld equation follows by making the identifications

$$k = \theta_x\ , \qquad \omega = -\theta_T\ .$$

The boundary conditions on $\psi^{(1)}$ are that $\psi^{(1)} = \psi^{(1)\prime} = 0$ at $y = 0,\omega$. When (2.11) is solved subject to these

boundary conditions, there results an eigenvalue condition of the form

$$f(\theta_X, \theta_T, X) = 0 \tag{2.12}$$

that must be satisfied for a nontrivial solution to (2.11) to exist. At each position X, we can calculate θ_X given θ_T or vice versa from (2.12). If θ_X is assumed given and real, then (2.12) yields a temporal stability analysis; if θ_T is assumed given and real, then a spatial stability analysis results. In either case, it is assumed that conditions are close to neutral so that any weak amplification or damping can be incorporated later into the evolution equations.

Eqs. (2.6-7) are inhomogeneous problems that can be solved at each location X. The more interesting problem arises with Eqs. (2.8-10) since they are each of the form

$$\mathcal{L}\psi = K F + G \tag{2.13}$$

Here $K(X)$ is either λ, α_1, or α_2 in (2.4), and F and G are known functions. In order to determine K, we invoke the Fredholm alternative so that

$$K = -\frac{\int G \chi \, dy}{\int F \chi \, dy} \tag{2.14}$$

where χ is the solution of the homogeneous adjoint problem

$$\mathcal{M}\chi = 0 \tag{2.15}$$

That is, $\mathcal{M}$ is the adjoint of $\mathcal{L}$ and the boundary conditions for χ are adjoint to those for ψ.

By this technique, all the functions appearing in (2.4) are found and hence the evolution of the amplitude A can be determined.

3. THREE-DIMENSIONAL NONLINEAR NONPARALLEL STABILITY THEORY OF COMPRESSIBLE BOUNDARY LAYER FLOWS

The equations governing the motion of a compressible fluid are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) \tag{3.1}$$

$$\rho \frac{Du_i}{Dt} = \rho X_i + \frac{\partial T_{ij}}{\partial x_j} \tag{3.2}$$

$$\rho \frac{DE}{Dt} = \Phi - T \frac{\partial p}{\partial T} \operatorname{div} \vec{v} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) \tag{3.3}$$

$$p = F(\rho, T) \tag{3.4}$$

These equations are those of continuity, momentum, energy, and state, respectively. The notation is standard, viz. p, ρ, T, $\vec{v} = (u_1, u_2, u_3)$, $\vec{x} = (x_1, x_2, x_3)$, t, E, k, $\vec{X} = (X_1, X_2, X_3)$, T_{ij}, and Φ denote pressure, density, velocity, spatial coordinates, time, internal energy, coefficient of thermal conductivity, external force, stress tensor, and dissipation. In addition, we use the following subsidiary relationships:

$$T_{ij} = -\left(p + \frac{2}{3}\nu \frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.5}$$

$$E = c_v T \tag{3.6}$$

$$\Phi = \frac{1}{2}\nu e_{ij} e_{ij} - \frac{2}{3}\nu \Delta^2 \tag{3.7}$$

where

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{3.8}$$

$$\Delta = \text{div } \vec{v} \tag{3.9}$$

Here ν and c_v are the coefficient of viscosity and the specific heat at constant volume. It is further assumed that the three transfer coefficients k, ν, and c_v may be assumed functions of temperature T alone.

Armed with these basic equations, the aim is to develop a theory for the nonlinear stability of a basic three-dimensional boundary-layer flow. The boundary will be taken to be at $y = 0$ so that the basic flow will have slow variations in the $x = x_1$ and $z = x_3$ directions, together with possible slow temporal variations. This feature is most easily accounted for by the introduction of slow variables X, Z, and T' (do <u>not</u> confuse with T) where

$$(X,Z,T') = \mu(x,z,t) , \qquad \mu << 1$$

The small parameter μ is the usual one on which boundary layer theory is based.

The unperturbed boundary layer solution will be denoted by overbars. This zeroth order solution is taken as given. In addition, the first-order corrections to the basic state do influence the evolution of disturbances.

The basic state is then given by the velocity components

$$\bar{\mathbf{v}} = (\bar{u}(y,X,Z,T'),\ \bar{v}(y,X,Z,T'),\ \bar{w}(y,X,Z,T')) \tag{3.10}$$

and density, temperature, and pressure fields

$$\bar{\rho}(y,X,Z,T'),\ \bar{T}(y,X,Z,T'),\ \bar{p}(y,X,Z,T') \tag{3.11}$$

For the stability analysis, each function f is replaced by $\bar{f} + f$ and perturbation equations are then derived. Third-order terms in amplitude must be retained in the present theory. To be specific, we perform some preliminary calculations. For example, since

$$E = c_v(T)T, \tag{3.12}$$

then

$$\bar{E} = c_v(\bar{T})\bar{T}, \tag{3.13}$$

and therefore

$$E = c_v(\bar{T} + T)(\bar{T} + T) - c_v(\bar{T})\bar{T} . \tag{3.14}$$

Taylor expansion of this latter result gives

$$E = E_1T + E_2T^2 + E_3T^3 + \ldots \tag{3.15}$$

where

$$\begin{aligned} E_1 &= \bar{c}_v + \bar{c}_v'\bar{T} \\ E_2 &= \bar{c}_v' + \bar{c}_v''\bar{T}/2 \\ E_3 &= \bar{c}_v''/2 + \bar{c}_v'''\bar{T}/6. \end{aligned} \tag{3.16}$$

Here primes denote derivatives of $c_v(\bar{T}) = \bar{c}_v$.

In the equation of state for the basic flow

$$\bar{p} = F(\bar{\rho},\bar{T}) , \tag{3.17}$$

so that the perturbation pressure is given by

$$p = F(\bar{\rho}+\rho,\ \bar{T}+T) - F(\bar{\rho},\bar{T}). \tag{3.18}$$

Taylor expanding, it follows that

$$p = F_{10}\rho + F_{01}T + \frac{1}{2}[F_{20}\rho^2 + F_{11}\rho T + F_{02}T^2]$$

$$+ \frac{1}{6}[F_{30}\rho^3 + F_{21}\rho^2 T + F_{12}\rho T^2 + F_{03}T^3] \tag{3.19}$$

where $F_{mn} = \partial^{m+n} F(\bar{\rho},\bar{T})/\partial\bar{\rho}^m\,\bar{T}^n$

In a similar way, the stress tensor is expanded as

$$T_{ij} = -\delta_{ij}[p + \frac{2}{3}\{(\bar{\mu}\frac{\partial u_k}{\partial x_k} + \bar{\mu}'T\frac{\partial \bar{u}_k}{\partial x_k}) + (\bar{\mu}'T\frac{\partial u_k}{\partial x_k} + \bar{\mu}''\frac{T^2}{2}\frac{\partial \bar{u}_k}{\partial x_k})$$

$$+ (\bar{\mu}''\frac{T^2}{2}\frac{\partial u_k}{\partial x_k} + \frac{\bar{\mu}'''T^3}{6}\frac{\partial \bar{u}_k}{\partial x_k})\}]$$

$$+ [\{\bar{\mu}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + T\bar{\mu}'(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i})\}$$

$$+ \{\bar{\mu}'T(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{\bar{\mu}''}{2}T^2(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i})\}$$

$$+ \{\bar{\mu}''\frac{T^2}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{\bar{\mu}'''}{6}T^3(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i})\}] . \tag{3.20}$$

and the dissipation function Φ is expanded as

$$\bar{\phi} = \{\bar{\mu}[\bar{e}_{\alpha\beta}e_{\alpha\beta} - \tfrac{4}{3}(\operatorname{div}\bar{\underline{v}})(\operatorname{div}\underline{v})] + \bar{\mu}'T[\tfrac{1}{2}\bar{e}^2_{\alpha\beta} - \tfrac{2}{3}(\operatorname{div}\bar{\underline{v}})^2]\}$$

$$+ \{\bar{\mu}[\tfrac{1}{2}e^2_{\alpha\beta} - \tfrac{2}{3}(\operatorname{div}\underline{v})^2] + \bar{\mu}'T[\bar{e}_{\alpha\beta}e_{\alpha\beta} - \tfrac{4}{3}(\operatorname{div}\bar{\underline{v}})(\operatorname{div}\underline{v})]$$

$$+ \frac{\bar{\mu}''T^2}{2}[\tfrac{1}{2}\bar{e}^2_{\alpha\beta} - \tfrac{2}{3}(\operatorname{div}\bar{\underline{v}})^2]\}$$

$$+ \{\bar{\mu}'T[\tfrac{1}{2}e^2_{\alpha\beta} - \tfrac{2}{3}(\operatorname{div}\underline{v})^2] + \frac{\bar{\mu}''T^2}{2}[\bar{e}_{\alpha\beta}e_{\alpha\beta} - \tfrac{4}{3}\operatorname{div}\bar{\underline{v}}\operatorname{div}\underline{v}]$$

$$+ \frac{\bar{\mu}'''T^3}{6}[\tfrac{1}{2}\bar{e}^2_{\alpha\beta} - \tfrac{2}{3}(\operatorname{div}\bar{\underline{v}})^2]\} \quad . \tag{3.21}$$

Other functions which occur in the fundamental equations can be expanded in this way and need not be recorded explicitly at this stage. Of more importance to our subsequent analysis is the fact that perturbation quantities, say g, will have both fast and slow space-time scale variations (see Sec. 2). In order to incorporate this feature, it is again desirable to introduce a phase function

$$\theta = \frac{\theta(X,Z,T')}{\mu} \tag{3.22}$$

so the perturbed quantity g is expressed in the form

$$g(y,\theta,X,Z,T') \tag{3.23}$$

It follows that derivatives of g depend on both the slow and fast variables. For example,

$$\frac{\partial g}{\partial x} = \theta_x g_\theta + \mu g_x \tag{3.24}$$

$$\frac{\partial^2 g}{\partial x^2} = \theta_x^2 g_{\theta\theta} + \mu(2\theta_x g_{\theta x} + \theta_{xx} g_\theta) + \mu^2 g_{xx} \tag{3.25}$$

Other derivatives can be calculated in a similar manner.

Each perturbation function is expanded and the nonlinear parameter ε is introduced to order products of perturbations. The two small parameters μ and ε are used to order the non-parallel and nonlinear effects, respectively. In order for these effects to be in balance, the choice $\mu = \varepsilon^2$ turns out to be the appropriate one.

The equations are now rewritten in this notation and only the terms consistent to the order of the calculation are retained.

For the continuity equation, we obtain

$$\theta_T\rho_\theta + \theta_x(\bar{u}\rho_\theta + \bar{\rho}u_\theta) + \bar{\rho}_y v + \bar{\rho}v_y + \theta_z(\bar{w}\rho_\theta + \bar{\rho}w_\theta)$$

$$+ \mu[\rho_T + (\rho\bar{u} + \bar{\rho}u)_x + (\rho\bar{v})_y + (\rho\bar{w} + \bar{\rho}w)_z]$$

$$+ \varepsilon[\theta_x(\rho u)_\theta + (\rho v)_y + \theta_z(\rho w)_\theta] + \varepsilon\mu[(\rho u)_x + (\rho w)_z] = 0 \quad (3.26)$$

For the x momentum equation,

$$\bar{\rho}[\theta_T + (\theta_x\bar{u} + \theta_z\bar{w})]u_\theta + \bar{\rho}\frac{\partial\bar{u}}{\partial y}v$$

$$+ \mu[\bar{\rho}(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z})u + \bar{\rho}\frac{\partial\bar{u}}{\partial x}u + \bar{\rho}\frac{\partial\bar{u}}{\partial z}w$$

$$+ (\bar{u}_T + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z)\rho]$$

$$+ \varepsilon[\bar{\rho}(\theta_x uu_\theta + vv_y + \theta_z wu_\theta) + \theta_T\rho u_\theta + \frac{\partial\bar{u}}{\partial y}\rho v + \theta_x\bar{u}\rho u_\theta + \theta_z\bar{w}\rho u_\theta]$$

$$+ \varepsilon^2[\rho(\theta_x u + \theta_z w)u_\theta + \rho v u_y]$$

$$= [-\theta_x p_\theta - \mu p_x - \frac{2}{3}\bar{\nu}\theta_x(\theta_x u_{\theta\theta} + \theta_z w_{\theta\theta} + v_{y\theta})$$

$$- \frac{2}{3}\bar{\nu}\mu[(\theta_x u_{\theta x} + \theta_z w_{\theta x} + v_{yx}) + \theta_{xx}u_\theta + \theta_{xz}w_\theta]$$

$$- \frac{2}{3}\bar{\nu}'\mu\bar{T}_x(\theta_x u_\theta + \theta_z w_\theta + v_y) + \bar{\nu}2\theta_x^2 u_{\theta\theta}$$

$$+ \mu\bar{\nu}(2)(\theta_x u_{\theta x} + \theta_{xx}u_\theta) + \mu 2\bar{\nu}'\bar{T}_x\theta_x u_\theta$$

$$+ \mu[-\frac{2}{3}\bar{\nu}(u_{\theta x} + w_{\theta z})\theta_x - \frac{2}{3}\bar{\nu}'T_\theta(\bar{u}_x + \bar{v}_y + \bar{w}_z)\theta_x$$

$$+ 2\bar{\nu}\theta_x u_{\theta x} + 2\bar{\nu}'\theta_x\bar{u}_x T_\theta]$$

$$+ \varepsilon[-\frac{2}{3}\bar{\nu}'\theta_x(T(\theta_x u_\theta + \theta_z w_\theta + v_y))_\theta + 2\bar{\nu}'\theta_x^2 (Tu_\theta)_\theta]$$

$$+ \varepsilon^2[-\frac{1}{3}\bar{\nu}''\theta_x\{T^2(\theta_x u_\theta + \theta_z w_\theta + v_y)\}_\theta + \bar{\nu}''\theta_x^2(T^2 u_\theta)_\theta]$$

$$+ [\bar{\nu}(u_{yy} + \theta_x v_{y\theta}) + \bar{\nu}'\bar{T}_y(u_y + \theta_x v_\theta) + \bar{\nu}'\frac{\partial^2\bar{u}}{\partial y^2}T$$

$$+ \bar{\nu}'\bar{u}_y T_y + \bar{\nu}''\bar{T}_y\bar{u}_y T]$$

$$+ \mu[\bar{\nu}v_{xy} + \bar{\nu}'T_y v_x]$$

$$+ \varepsilon\{\bar{\nu}'T(u_y + \theta_x v_\theta) + \frac{\bar{\nu}''}{2}\bar{u}_y T^2\}_y$$

$$+ \varepsilon^2[\frac{\bar{\nu}''}{2}T^2(u_y + \theta_x v_\theta) + \frac{\bar{\nu}'''}{6}\bar{u}_y T^3]_y$$

$$+ \bar{\nu}\,\theta_z(\theta_x w_{\theta\theta} + \theta_z u_{\theta\theta})$$

$$+ \mu[\bar{\nu}(\theta_x w_{\theta z} + \theta_z u_{\theta z} + \theta_{zx} w_\theta + \theta_{zz} u_\theta) + \bar{\nu}\,\bar{T}_z(\theta_x w_\theta + \theta_z u_\theta)$$

$$+ \bar{\nu}(w_{x\theta} + u_{z\theta})\theta_z + \bar{\nu}'\theta_z T_\theta(\bar{u}_z + \bar{w}_x)]$$

$$+ \varepsilon\theta_z[\bar{\nu}'T(\theta_x w_\theta + \theta_z u_\theta)]_\theta$$

$$+ \varepsilon^2\theta_z[\bar{\nu}''\frac{T^2}{2}(\theta_x w_\theta + \theta_z u_\theta)]_\theta \qquad (3.27)$$

The y momentum perturbation equation is

$$\bar{\rho}[\theta_T + (\theta_x\bar{u} + \theta_z\bar{w})]v_\theta$$

$$+ \mu[\bar{\rho}(v_T + (\bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z})v + \bar{\rho}\frac{\partial\bar{v}}{\partial y}v]$$

$$+ \varepsilon[\bar{\rho}((\theta_x u + \theta_z w)v_\theta + vv_y) + \theta_T\rho v_\theta]$$

$$+ \varepsilon^2[\rho((u\theta_x + w\theta_z)v_\theta + vv_y)]$$

$$= -p_y - \frac{2}{3}\bar{\nu}(\theta_x u_{\theta y} + \theta_z w_{\theta y} + v_{yy})$$

$$- \frac{2}{3}\bar{\nu}'\bar{T}_y(\theta_x u_\theta + \theta_z w_\theta + v_y) + 2\bar{\nu}v_{yy} + 2\bar{\nu}'\bar{T}_y v_y]$$

$$+ \mu[-\frac{2}{3}\bar{\nu}(u_x + w_z) - \frac{2}{3}\bar{\nu}'(\bar{u}_x + \bar{v}_y + \bar{w}_z)T + 2\bar{\nu}'\bar{v}_y T]_y$$

$$+ \varepsilon[-\frac{2}{3}\bar{\nu}'T(\theta_x u_\theta + \theta_z w_\theta + v_y) + 2\bar{\nu}'Tv_y]_y$$

$$+ \varepsilon^2[-\frac{1}{3}\bar{\nu}''T^2(\theta_x u_\theta + \theta_z w_\theta + v_y) + \bar{\nu}'T^2 v_y]_y$$

$$+ \theta_z[\bar{\nu}(w_{y\theta} + \theta_z v_{\theta\theta}) + \bar{\nu}'\bar{w}_y T_\theta]$$

$$+ \mu[\bar{\nu}(w_{yz} + \theta_z v_{\theta z} + \theta_{zz} v_\theta) + \bar{\nu}'\bar{T}_z(w_y + \theta_z v_\theta)$$

$$+ \bar{\nu}'\bar{w}_y T_z + \bar{\nu}'\bar{w}_{yz}T + \bar{\nu}''\bar{T}_z\bar{w}_y T + \theta_z\bar{\nu}v_{\theta z}]$$

$$+ \epsilon\theta_z[\bar{\nu}'T(w_y + \theta_z v_\theta) + \frac{\bar{\nu}''}{2}\bar{w}_y T^2]_\theta$$

$$+ \epsilon^2\theta_z[\frac{\bar{\nu}''T^2}{2}(w_y + \theta_z v_\theta) + \frac{\bar{\nu}'''}{6}\bar{w}_y T^3]_\theta$$

$$+ \theta_x[\bar{\nu}(u_{y\theta} + \theta_x v_{\theta\theta}) + \bar{\nu}'\bar{u}_y T_\theta]$$

$$+ \mu[\bar{\nu}(u_{yx} + \theta_x v_{\theta x} + \theta_{xx} v_\theta) + \bar{\nu}'\bar{T}_x(u_y + \theta_x v_\theta)$$

$$+ \bar{\nu}'\bar{u}_y T_x + \bar{\nu}'\bar{u}_{yx}T + \bar{\nu}'\bar{T}_x\bar{u}_y T + \theta_x\bar{\nu}v_{\theta x}]$$

$$+ \epsilon\theta_x[\bar{\nu}T(u_y + \theta_x v_\theta) + \frac{\bar{\nu}''}{2}\bar{u}_y T^2]_\theta$$

$$+ \epsilon^2\theta_x[\bar{\nu}''\frac{T^2}{2}(u_y + \theta_x v_\theta) + \frac{\bar{\nu}'''}{6}\bar{u}_y T^3]_\theta \quad . \qquad (3.28)$$

The z momentum equation is analogous to the corresponding x equation and takes the form

$$\bar{\rho}[\theta_T + (\theta_x\bar{u} + \theta_z\bar{w})]w_\theta + \bar{\rho}\,\bar{w}_y\,v$$

$$+ \mu[\bar{\rho}(\frac{\partial}{\partial T} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z})w + \bar{\rho}\frac{\partial\bar{w}}{\partial z}w + \bar{\rho}\frac{\partial\bar{w}}{\partial x}u$$

$$+ \bar{\rho}(\bar{w}_T + \bar{u}\bar{w}_x + \bar{v}\bar{w}_y + \bar{w}\bar{w}_z)]$$

$$+ \varepsilon[\bar{\rho}(\theta_x uw_\theta + vw_y + \theta_z ww_\theta) + \theta_T\rho w_\theta + \frac{\partial\bar{w}}{\partial y}\rho v + \theta_x\bar{u}\rho w_\theta + \theta_z\bar{w}\rho w_\theta]$$

$$+ \varepsilon^2[\rho(\theta_x u + \theta_z w)w_\theta + \rho v w_y]$$

$$= \bar{\nu}\theta_x(\theta_z u_{\theta\theta} + \theta_x w_{\theta\theta})$$

$$+ \mu[\bar{\nu}(\theta_z u_{\theta x} + \theta_x w_{\theta x} + \theta_{zx}u_\theta + \theta_{xx}w_\theta) + \bar{\nu}'\bar{T}_x(\theta_z u_\theta + \theta_x w_\theta)$$

$$+ \bar{\nu}(u_{z\theta} + w_{x\theta})\theta_x + \bar{\nu}'\theta_x T_\theta(\bar{u}_z + \bar{w}_x)]$$

$$+ \varepsilon\theta_x[\bar{\nu}'T(\theta_z u_\theta + \theta_x w_\theta)]_\theta$$

$$+ \varepsilon^2\theta_x[\frac{\bar{\nu}''T^2}{2}(\theta_z u_\theta + \theta_x w_\theta)]_\theta$$

$$+ \bar{\nu}(w_{yy} + \theta_z v_{\theta y}) + \bar{\nu}'\bar{T}_y(w_y + \theta_z v_\theta) + \bar{\nu}'\bar{w}_{yy}T_y$$

$$+ \bar{\nu}''\bar{T}_y\bar{w}_y T + \bar{\nu}'\bar{w}_{yy}T$$

$$
\begin{aligned}
&+ \mu(\bar{\nu} v_{yz} + \bar{\nu}' \bar{T}_y v_z) \\
&+ \varepsilon[\bar{\nu}' T(w_y + \theta_z v_\theta) + \frac{\bar{\nu}''}{2} \bar{w}_y T^2]_y \\
&+ \varepsilon^2[\frac{\bar{\nu}''}{2} T^2(w_y + \theta_z v_\theta) + \frac{\bar{\nu}'''}{6} \bar{w}_y T^3]_y \\
&+ [-p_\theta \theta_z - \frac{2}{3} \bar{\nu} \theta_z(\theta_x u_{\theta\theta} + \theta_z w_{\theta\theta} + v_{y\theta}) + 2\bar{\nu}\theta_z^2 w_{\theta\theta}] \\
&+ \mu[-p_z - \frac{2}{3} \bar{\nu}(\theta_x u_{\theta z} + \theta_z w_{\theta z} + v_{yz} + \theta_{xz} u_\theta + \theta_{zz} w_\theta) \\
&\qquad - \frac{2}{3} \bar{\nu}' \bar{T}_z(\theta_x u_\theta + \theta_z w_\theta + v_y) + 2\bar{\nu}\theta_z w_{\theta z} + (\bar{\nu} v_z)_y \\
&\qquad + 2\bar{\nu}\theta_{zz} w_\theta + 2\bar{\nu}' \bar{T}_z \theta_z w_\theta \\
&\qquad - \frac{2}{3} \theta_z \bar{\nu}(u_{\theta x} + w_{\theta z}) - \frac{2}{3} \bar{\nu}' \theta_z T_\theta(\bar{u}_x + \bar{v}_y + \bar{w}_z) \\
&\qquad + \theta_z 2\bar{\nu} w_{\theta z} + 2\bar{\nu}' \bar{w}_z \theta_z T_\theta] \\
&+ \varepsilon\theta_z[-\frac{2}{3} \bar{\nu}' T(\theta_x u_\theta + \theta_z w_\theta + v_y) + 2\bar{\nu}' \theta_z T w_\theta]_\theta \\
&+ \varepsilon^2 \theta_z[-\frac{1}{3} \bar{\nu}'' T^2(\theta_x u_\theta + \theta_z w_\theta + v_y) + \bar{\nu}'' \theta_z T^2 w_\theta]_\theta \quad . \qquad (3.29)
\end{aligned}
$$

The energy equation is

$$\bar{\rho}\bar{c}_v[\theta_T T_\theta + \bar{u}\theta_x T_\theta + \bar{w}\theta_z T_\theta + \bar{T}_y v]$$

$$+ \mu[\bar{\rho}\bar{c}_v\{T_T + \bar{u}T_x + \bar{v}T_y + \bar{w}T_z + u\bar{T}_x + w\bar{T}_z\}]$$

$$+ \varepsilon[\bar{\rho}\bar{c}_v\{\theta_x uT_\theta + vT_y + \theta_z wT_\theta\}$$

$$+ (\bar{c}_v\rho + \bar{\rho}\bar{c}_v'T)(\theta_T T_\theta + \bar{u}\theta_x T_\theta + \bar{w}\theta_z T_\theta + \bar{T}_y v)]$$

$$+ \varepsilon^2[(\bar{c}_v\rho + \bar{\rho}\bar{c}_v'T)(\theta_x uT_\theta + vT_y + \theta_z wT_\theta)$$

$$+ (\bar{c}_v\rho T + \bar{\rho}\frac{\bar{c}_v''}{2}T^2)(\theta_T T_\theta + \theta_x\bar{u}T_\theta + \theta_z\bar{w}T_\theta + \bar{T}_y v)]$$

$$= \bar{\nu}[2\bar{u}_y(u_y + \theta_x v_\theta) + 2\bar{w}_y(w_y + \theta_z v_\theta)] + \bar{\nu}'(\bar{u}_y^2 + \bar{w}_y^2)T$$

$$+ \mu[\bar{\nu}[2\bar{u}_y v_x + 2\bar{w}_y v_z + 2(\bar{u}_z + \bar{w}_x)(\theta_z u_\theta + \theta_x w_\theta)$$

$$+ \frac{4}{3}\{2\bar{u}_x\theta_x u_\theta + 2\bar{v}_y v_y + 2\bar{w}_z\theta_z w_\theta$$

$$- (\bar{u}_x v_y + \bar{v}_y\theta_x u_\theta) - (\bar{w}_z v_y + \bar{v}_y\theta_z w_\theta)\}]$$

$$+ \varepsilon[\bar{\nu}\{(u_y + \theta_x v_\theta)^2 + (w_y + \theta_z v_\theta)^2 + (\theta_z u_\theta + \theta_x w_\theta)^2\}$$

$$+ \frac{4}{3}\bar{\nu}\{\theta_x^2 u_\theta^2 + v_y^2 + \theta_z^2 w_\theta^2 - (\theta_x u_\theta + \theta_z w_\theta)v_y - \theta_x\theta_z u_\theta w_\theta\}$$

$$+ \bar{\nu}'T\{2\bar{u}_y(u_y + \theta_x v_\theta) + 2\bar{w}_y(w_y + \theta_z v_\theta)\} + \frac{\bar{\nu}''}{2}T^2(\bar{u}_y^2 + \bar{w}_y^2)]$$

$$+ \varepsilon^2[\bar{\nu}'T\{(u_y + \theta_x v_\theta)^2 + (w_y + \theta_z v_\theta)^2 + (\theta_z u_\theta + \theta_x w_\theta)^2\}$$

$$+ \frac{4}{3}\bar{\nu}'T\{\theta_x^2 u_\theta^2 + v_y^2 + \theta_z^2 w_\theta^2 - (\theta_x u_\theta + \theta_z w_\theta)v_y - \theta_x\theta_z u_\theta w_\theta\}]$$

$$+ \frac{\bar{\nu}''T^2}{2}\{2\bar{u}_y(u_y + \theta_x v_\theta) + 2\bar{w}_y(w_y + \theta_z v_\theta)\}$$

$$+ \frac{\bar{\nu}'''T^3}{6}[\bar{u}_y^2 + \bar{w}_y^2]$$

$$- \bar{T}F_{01}(\theta_x u_\theta + v_y + \theta_z w_\theta)$$

$$+ \mu[-\bar{T}(\bar{u}_x + \bar{v}_y + \bar{w}_z)(F_{11}\rho + F_{02}T) - \bar{T}(u_x + w_z)F_{01}$$

$$- TF_{01}(\bar{u}_x + \bar{v}_y + \bar{w}_z)]$$

$$+ \varepsilon[-\bar{T}(\theta_x u_\theta + \theta_z w_\theta + v_y)(\rho F_{11} + TF_{02}) - T(\theta_x u_\theta + \theta_z w_\theta + v_y)F_{01}]$$

$$+ \varepsilon^2[-\bar{T}(\theta_x u_\theta + \theta_z w_\theta + v_y)(\frac{\rho^2}{2}F_{21} + \rho TF_{22} + F_{03}\frac{T^2}{2})$$

$$- T(\theta_x u_\theta + \theta_z w_\theta + v_y)(\rho F_{11} + TF_{02})]$$

$$+ \bar{k}\theta_x(\theta_x T_{\theta\theta}) + \bar{k}\theta_z(\theta_z)T_{\theta\theta} + \frac{\partial}{\partial y}(\bar{k}T_y)$$

$$+ \mu[(\bar{k}\theta_x T_\theta)_x + \theta_x\bar{k}T_{x\theta} + \bar{k}'\bar{T}_x\theta_x T_\theta + (\bar{k}\theta_z T_\theta)_z$$

$$+ \theta_z\bar{k}T_{z\theta} + \bar{k}'T_z\theta_z T_\theta]$$

$$+ \epsilon[\bar{k}'\theta_x^2\,(TT_\theta)_\theta + \bar{k}'\theta_z^2\,(TT_\theta)_\theta + (\bar{k}'TT_y)_y]$$

$$+ \epsilon^2[\frac{\bar{k}''}{9}\,(\theta_x^2 + \theta_z^2)(T^2T_\theta)_\theta + (\frac{\bar{k}''T^3}{6})_{yy}] \quad . \tag{3.30}$$

Finally, the equation of state is

$$= P_{10} + F_{01}T + \epsilon\,[\tfrac{1}{2}F_{20}\rho^2 + F_{11}\rho T + \tfrac{1}{2}F_{02}T^2]$$

$$+ \epsilon^2[\tfrac{1}{6}F_{30}\rho^3 + \tfrac{1}{2}F_{21}\rho^2T + \tfrac{1}{2}F_{12}\rho T^2 + \tfrac{1}{6}F_{03}T^3] \quad . \tag{3.31}$$

The perturbation equations (3.26-31) are to be solved subject to appropriate boundary conditions. At $y = \infty$, it is clear that all perturbations should decay to 0 so we require

$$(\rho,u,v,w,p,T) \to 0 \qquad (y \to \infty)$$

At the rigid boundary $y = 0$, $u = v = w = 0$ and the temperature condition will be of the form

$$\alpha T + \beta \frac{\partial T}{\partial y} = 0$$

The case $\beta = 0$ corresponds to an isothermal wall and the case $\alpha = 0$ corresponds to an adiabatic wall.

4. FINAL EQUATIONS FOR THREE-DIMENSIONAL NONLINEAR NONPARALLEL STABILITY THEORY OF COMPRESSIBLE BOUNDARY LAYER FLOWS

First consider the linear problem so that $\varepsilon = \mu = 0$. We ask for solutions in which each amplitude function is written in the form

$$h(\theta,y,X,Z,T') = Ah^{(1)}(y,X,Z,T')e^{i\theta} \tag{4.1}$$

where the envelope amplitude is to be determined as a function of the slow variables, i.e.,

$$A = A(X,Z,T') \tag{4.2}$$

The respective equations are:

$$i(\theta_{T'} + \bar{u}\theta_X + \bar{w}\theta_Z)\rho^{(1)} + i\theta_X \bar{\rho} u^{(1)} + i\theta_Z \bar{\rho} w^{(1)} + (\bar{\rho} v^{(1)})_y = 0 \tag{4.3}$$

$$\begin{aligned} &i\bar{\rho}(\theta_{T'} + \theta_X\bar{u} + \theta_Z\bar{w})u^{(1)} + \bar{\rho}\bar{u}_y v^{(1)} + i\theta_X p^{(1)} \\ &- \tfrac{2}{3}\bar{\gamma}\theta_X(\theta_X u^{(1)} + \theta_Z w^{(1)} - iv_y^{(1)}) + 2\bar{\gamma}\theta_X^2 u^{(1)} \\ &+ \bar{\gamma}\theta_Z(\theta_Z u^{(1)} + \theta_X w^{(1)}) - (\bar{\gamma}(u_y^{(1)} + i\theta_X v^{(1)})_y \\ &- (\bar{\gamma}'\bar{u}_y T^{(1)})_y = 0 . \end{aligned} \tag{4.4}$$

$$i\bar{\rho}(\theta_{T'} + \theta_x\bar{u} + \theta_z\bar{w})v^{(1)} + p_y^{(1)} - i\bar{\nu}\theta_x(u_y^{(1)} + i\theta_x v^{(1)})$$

$$- i(\theta_x\bar{\nu}'\bar{u}_y + \theta_z\bar{\nu}'\bar{w}_y)T^{(1)} - i\bar{\gamma}\theta_z(w_y^{(1)} + i\theta_z v^{(1)})$$

$$- 2(\bar{\gamma}v_y^{(1)})_y + \frac{2}{3}\left[\bar{\nu}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})\right] = 0 \qquad (4.5)$$

$$i\bar{\rho}(\theta_{T'} + \theta_x\bar{u} + \theta_z\bar{w})w^{(1)} + \bar{\rho}\bar{w}_y v^{(1)} + i\theta_z p^{(1)}$$

$$- \frac{2}{3}\bar{\nu}\theta_z(\theta_x u^{(1)} + \theta_z w^{(1)} - iv_y^{(1)})$$

$$- \left[\bar{\gamma}(w_y^{(1)} + i\theta_z v^{(1)})\right]_y + 2\bar{\gamma}\theta_z w^{(1)} - (\bar{\gamma}'\bar{w}_y T^{(1)})_y$$

$$- \left[\bar{\gamma}(w_y^{(1)} + i\theta_z v^{(1)})\right]_y + \bar{\gamma}\theta_x(\theta_z u^{(1)} + \theta_x w^{(1)}) = 0 \qquad (4.6)$$

$$\bar{\rho}\bar{c}_v\left[i(\theta_{T'} + \bar{u}\theta_x + \bar{w}\theta_z)T^{(1)} + \bar{T}_y v^{(1)}\right]$$

$$- \bar{\gamma}\{2\bar{u}_y(u_y^{(1)} + i\theta_x v^{(1)}) + 2\bar{w}_y(w_y^{(1)} + i\theta_z v^{(1)})\}$$

$$- \bar{\gamma}'(\bar{u}_y^2 + \bar{w}_y^2)T^{(1)} - F_{01}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})T^{(1)}$$

$$- (\bar{k}T_y^{(1)})_y + \bar{k}(\theta_x^2 + \theta_z^2)T^{(1)} = 0 \qquad (4.7)$$

$$p^{(1)} - F_{10}\rho^{(1)} - F_{01}T^{(1)} = 0 . \qquad (4.8)$$

The above set of equations (4.3-8) constitutes the standard eigenvalue problem for determination of the linear stability of a given boundary layer configuration. Rather than eliminate variables it is convenient to consider this system of equations to be written in matrix form

$$(\underline{A}_1 \frac{d^2}{dy^2} + \underline{B}_1 \frac{d}{dy} + \underline{C}_1)\underset{\sim}{H}^{(1)} = 0 \qquad (4.9)$$

where $\underset{\sim}{H}^{(1)} = (\rho^{(1)}, u^{(1)}, v^{(1)}, w^{(1)}, T^{(1)}, p^{(1)})$ and the 6×6 matrices $\underline{A}_1$, $\underline{B}_1$, $\underline{C}_1$ are readily written down from the previous equation.

The solution of the linear eigenvalue problem gives a relationship between $-\theta_{T'}$, θ_X, and θ_Z at each position and time X,Z,T'. Of course the quantities $-\theta_{T'}$, θ_X and θ_Z are the local frequency and X and Z wave numbers. At this stage we make no distinction between temporal and spatial amplification. In actual computation where the initial instability is being followed and we are close to neutral conditions the variable θ is treated to be approximately real. Note that at this stage of the calculation the wave packet amplitude $A(X,Z,T')$ is arbitrary.

In order to proceed into the nonlinear problem it is necessary to return to (4.1) and replace this equation for any perturbation function by an expansion of the form

$$
\begin{aligned}
h = {} & Ah^{(1)}e^{i\theta} + A^*h^{(1)*}e^{-i\theta} \\
& + \varepsilon(A^2h^{(2)}e^{2i\theta} + A^{*2}h^{(2)*}e^{-2i\theta} + AA^*h^{(0)}) \\
& + \mu\Big[(A_X h_1^{(1)} + A_Z h_2^{(1)} + Ah_3^{(1)})e^{i\theta} \\
& \qquad + (A_X^* h_1^{(1)*} + A_Z^* h_2^{(1)*} + A^* h_3^{(1)*})e^{-i\theta}\Big] \\
& + \varepsilon^2\Big[A^2A^*h^{(11)}e^{i\theta} + A^*A^2h^{(11)*}e^{-i\theta} \\
& \qquad + A^3h^{(3)}e^{3i\theta} + A^{*3}h^{(3)*}e^{-3i\theta}\Big] \\
& + O(\varepsilon\mu, \mu^2, \varepsilon^3) \quad . \qquad (4.10)
\end{aligned}
$$

This expansion for any function h can be replaced by the identical expansion for the vector

$$\underset{\sim}{H} = (\rho, u, v, w, T, p) \quad , \qquad (4.11)$$

where $\underset{\sim}{H}^{(1)}$ is defined in (4.9).

The expansion and truncation procedure is straightforward and based on the fact that μ and ε^2 are of the same order. The process necessitates that a corresponding expansion be invoked for $A(X,Z,T')$ of the form

$$\mu A_{T'} = \mu(\alpha_1 A_X + \alpha_2 A_Z + \alpha_3 A) + \varepsilon^2 \lambda A^2 A^* \quad , \qquad (4.12)$$

where α_1, α_2, α_3, and λ are scalar functions of X, Z, and T'. These functions in this evolution equation

are determined by orthogonality conditions associated with the other boundary value problems. In particular those for $\underset{\sim}{H}_1^{(1)}$, $\underset{\sim}{H}_2^{(1)}$, $\underset{\sim}{H}_3^{(1)}$, and $\underset{\sim}{H}^{(11)}$ are connected to these functions and in addition the non-resonant boundary value problems for $\underset{\sim}{H}^{(2)}$ and $\underset{\sim}{H}^{(0)}$ are needed.

First we write down the inhomogeneous problem for $\underset{\sim}{H}^{(2)}$, namely

$$(\underset{\sim}{A}_2 \frac{d^2}{dy^2} + \underset{\sim}{B}_2 \frac{d}{dy} + \underset{\sim}{C}_2)\underset{\sim}{H}^{(2)} + \underset{\sim}{K}^{(2)} = 0 \tag{4.13}$$

Here the solution used is that

$$\underset{\sim}{H}^{(2)} = (\rho^{(2)}, u^{(2)}, v^{(2)}, w^{(2)}, T^{(2)}, p^{(2)}) \tag{4.14}$$

while the matrices $\underset{\sim}{A}_2$, $\underset{\sim}{B}_2$, and $\underset{\sim}{C}_2$ are identical to those for $\underset{\sim}{A}_1$, $\underset{\sim}{B}_1$, and $\underset{\sim}{C}_1$ except that the variable θ is everywhere replaced by 2θ. The vector $\underset{\sim}{K}^{(2)}$ is readily found and if

$$\underset{\sim}{K}^{(2)} = (k_1^{(2)}, k_2^{(2)}, k_3^{(2)}, k_4^{(2)}, k_5^{(2)}, k_6^{(2)}) , \tag{4.15}$$

we find that

$$k_1^{(2)} = 2i\theta_x \rho^{(1)} u^{(1)} + 2i\theta_z \rho^{(1)} w^{(1)} + (\rho^{(1)} v^{(1)})_y , \tag{4.16}$$

$$k_2^{(2)} = \Big[\bar{\rho}(i\theta_x u^{(1)2} + v^{(1)}u_y^{(1)} + i\theta_z w^{(1)}u^{(1)})$$

$$+ \rho^{(1)}(i(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z)u^{(1)} + \bar{u}_y v^{(1)})\Big]$$

$$+ \tfrac{2}{3}\bar{\gamma}'\theta_x\, 2iT^{(1)}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$- 2\bar{\gamma}'\theta_x^2\, 2iiT^{(1)}u^{(1)}$$

$$- \{\bar{\gamma}'T^{(1)}(u_y^{(1)} + i\theta_x v^{(1)}) + \bar{\gamma}\tfrac{n}{2}\,\bar{u}_y T^{(1)2}\}_y$$

$$- 2i\bar{\gamma}'\theta_z T^{(1)}i(\theta_z u^{(1)} + \theta_x w^{(1)}) \,. \qquad (4.17)$$

$$k_3^{(2)} = \bar{\rho}\Big[(i\theta_x u^{(1)}v^{(1)} + i\theta_z w^{(i)}v^{(1)} + v^{(1)}v_y^{(1)}) + i\theta_T,\rho^{(1)}v^{(1)}\Big]$$

$$- 2i\theta_x\{\bar{\gamma}'T^{(1)}(u_y^{(1)} + i\theta_x v^{(1)}) + \tfrac{1}{2}\bar{\gamma}''u_y T^{(1)2}\}$$

$$+ \{\tfrac{2}{3}\gamma'T^{(1)}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)}) - 2\bar{\gamma}'T^{(1)}v_y^{(1)}\}_y$$

$$- 2i\theta_z\{\bar{\gamma}'T^{(1)}(w_y^{(1)} + i\theta_z v^{(1)}) + \tfrac{1}{2}\bar{\gamma}''w_y T^{(1)2}\} \qquad (4.18)$$

$$k_4^{(2)} = \Big[\bar{\rho}(i\theta_x u^{(1)} w^{(1)} + v^{(1)} w_y^{(1)} + i\theta_z w^{(1)2})$$

$$+ \rho^{(1)}\, i(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z) w^{(1)} + \bar{w}_y v^{(1)}\Big]$$

$$- 2i\theta_x \bar{\gamma}' T^{(1)} i(\theta_z u^{(1)} + \theta_x w^{(1)})$$

$$- \{\bar{\gamma} T^{(1)} (w_y^{(1)} + i\theta_z v^{(1)}) + \tfrac{1}{2} \bar{\gamma}'' \bar{w}_y T^{(1)2}\}_y$$

$$- 2i\theta_z \{-\tfrac{2}{3} \bar{\gamma}' T^{(1)} (i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)})$$

$$+ 2\bar{\gamma}' \theta_z T^{(1)} i w^{(1)}\}_- \,, \qquad (4.19)$$

$$k_5^{(2)} = \Big[\overline{\rho c_v}(i\theta_x u^{(1)} T^{(1)} + v^{(1)} T_y^{(1)} + i\theta_z w^{(1)} T^{(1)})$$

$$+ (\bar{c}_v \rho^{(1)} + \bar{\rho}\, \bar{c}_v T^{(1)})(i(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z) T^{(1)} + \bar{T}_y v^{(1)})\Big]$$

$$- \bar{\gamma}\{(u_y^{(1)} + i\theta_x v^{(1)})^2 + (w_y^{(1)} + i\theta_z v^{(1)})^2$$

$$- (\theta_z u^{(1)} + \theta_x w^{(1)})^2$$

$$+ \tfrac{4}{3}\Big(-\theta_x^2 u^{(1)2} + v_y^{(1)2} - \theta_z^2 w^{(1)2}$$

$$- i(\theta_x u^{(1)} + \theta_z w^{(1)}) v_y^{(1)} + \theta_x \theta_z u^{(1)} w^{(1)}\}$$

$$- 2\bar{\gamma}' T^{(1)} \{\bar{u}_y (u_y^{(1)} + i\theta_x v^{(1)}) + \bar{w}_y (w_y^{(1)} + i\theta_z v^{(1)})\}$$

$$- \tfrac{1}{2} \bar{\gamma}'' T^{(1)2} (\bar{u}_y^2 + \bar{w}_y^2)$$

$$+ \bar{T}(i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)})(F_{11} \rho^{(1)} + F_{02} T^{(1)})$$

$$+ F_{01}T^{(1)}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$- \bar{k}'i(\theta_x^2 + \theta_z^2)T^{(1)2} - \bar{k}'(T^{(1)}T_y^{(1)})_y \quad , \tag{4.20}$$

$$k_6^{(2)} = -\tfrac{1}{2}F_{20}\rho^{(1)2} - F_{11}(\rho^{(1)}T^{(1)}) - \tfrac{1}{2}F_{02}T^{(1)2} \quad . \tag{4.21}$$

With these specifications and the appropriate boundary conditions the inhomogeneous problem defined by (4.13) can be solved.

In the same way

$$(\mathcal{A}_0 \frac{d^2}{dy^2} + \mathcal{B}_0 \frac{d}{dy} + \mathcal{C}_0)\mathcal{H}^{(0)} + \mathcal{K}^{(0)} = 0 \tag{4.22}$$

is the appropritate equation governing the induced second order mean motion. Here the matrices $\mathcal{A}^{(0)}$, $\mathcal{B}^{(0)}$, and $\mathcal{C}^{(0)}$ are those for $\mathcal{A}^{(1)}$, $\mathcal{B}^{(1)}$, and $\mathcal{C}^{(1)}$ except that θ is put equal to zero. If the vector $\mathcal{K}^{(0)}$ is written in the form

$$\mathcal{K}^{(0)} = (k_1^{(0)}, k_2^{(0)}, k_3^{(0)}, k_4^{(0)}, k_5^{(0)}, k_6^{(0)}) \quad , \tag{4.23}$$

then the individual components are given by

$$k_1^{(0)} = (\rho^{(1)}v^{(1)*} + \rho^{(1)*}v^{(1)})_y \quad , \tag{4.24}$$

$$k_2^{(0)} = \bar{\rho}\Big[v^{(1)}u_y^{*(1)} + v^{*(1)}u_y^{(1)} + \theta_z(w^{(1)} - iu^{(1)*}iu^{(1)})\Big]$$

$$+ (\theta_T + u\theta_x + w\theta_z)(i\rho^{(1)}u^{(1)*} - i\rho^{(1)*}u^{(1)})$$

$$+ \bar{u}_y(\rho^{(1)}v^{(1)*} + \rho^{(1)*}v^{(1)})$$

$$- \Big[\bar{\gamma}'\{T^{(1)}(u_y^{(1)*} - i\theta_x v^{(1)*}) + T^{(1)*}(u_y^{(1)} + i\theta_x v^{(1)})$$

$$+ \bar{\gamma}''\bar{u}_y T^{(1)}T^{(1)*}\}_y\Big] \qquad (4.25)$$

$$k_3^{(0)} = \rho\Big[\theta_x(u^{(1)*}iv^{(1)} - iu^{(1)}v^{(1)*})$$

$$+ \theta_z(w^{(1)*}iv^{(1)} - iw^{(1)}v^{(1)*}) + 2v^{(1)}v_y^{(1)*}\Big]$$

$$+ \theta_T(\rho^{(1)*}iv^{(1)} - i\rho^{(1)}v^{(1)*})$$

$$- \{-\tfrac{2}{3}\bar{\gamma}'\{T^{(1)}(-i\theta_x u^{(1)*} + v_y^{(1)*} - i\theta_z w^{(1)*})$$

$$+ T^{(1)*}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})\}$$

$$+ 2\bar{\gamma}'(T^{(1)}v_y^{(1)*} + T^{(1)*}v_y^{(1)})\}_y \ , \qquad (4.26)$$

$$k_4^{(0)} = \bar{\rho}\,\theta_x(iu^{(1)*}w^{(1)} - u^{(1)}w^{(1)*}) + (v^{(1)}w_y^{(1)*} + v^{(1)*}w_y^{(1)})$$

$$+ (\theta_T + \bar{u}\theta_x + \bar{w}\theta_z)(\rho^{(1)} - iw^{(1)*} + i\rho^{(1)*}w^{(1)})$$

$$+ \bar{w}_y(\rho^{(1)}v^{(1)*} + \rho^{(1)*}v^{(1)})$$

$$- \Big[\bar{\gamma}'\{T^{(1)}(w_y^{(1)*} - i\theta_z v^{(1)*}) + T^{(1)*}(w_y^{(1)} + i\theta_z v^{(1)})\}$$

$$+ \bar{\gamma}''\bar{w}_y T^{(1)}T^{(1)*}\Big]_y \ , \qquad (4.27)$$

$$
\begin{aligned}
k_5^{(0)} = {} & \bar{\rho}\bar{c}_v\Big[\theta_x i(u^{(1)*}T^{(1)} - u^{(1)}T^{(1)*}) + v^{(1)}T_y^{(1)*} + v^{(1)*}T_y^{(1)} \\
& + \theta_z i(w^{(1)*}T^{(1)} - w^{(1)}T^{(1)*})\Big] \\
& + \bar{c}_v(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z)(i)(\rho^{(1)*}T^{(1)} - \rho^{(1)}T^{(1)*}) \\
& + \bar{\rho}\bar{c}_v'\bar{T}_y(T^{(1)}v^{(1)*} + T^{(1)*}v^{(1)}) \\
& + \bar{c}_v\bar{T}_y(\rho^{(1)}v^{(1)*} + \rho^{(1)*}v^{(1)}) \\
& - \bar{\gamma}\Big[2u_y^{(1)}u_y^{(1)*} + 2\theta_x(u_y^{(1)} - iv^{(1)*} + u_y^{(1)*}iv^{(1)}) \\
& + (\theta_x^2 + \theta_z^2)\,2v^{(1)}v^{(1)*} + 2w_y^{(1)}w_y^{(1)*} \\
& + 2\theta_z(w_y^{(1)}(-i)v^{(1)*} + w_y^{(1)*}iv^{(1)}) \\
& + (\theta_z^2 + \tfrac{4}{3}\theta_x^2)\,2u^{(1)}u^{(1)*} + (\theta_x^2 + \tfrac{4}{3}\theta_z^2)\,2w^{(1)}w^{(1)*} \\
& - \tfrac{2}{3}\theta_x\theta_z\Big[iu^{(1)} - iw^{(1)*} - iu^{(1)*}iw^{(1)}\Big] \\
& + \tfrac{4}{3}\,2v_y^{(1)}v_y^{(1)*} - \tfrac{4}{3}\theta_x(iu^{(1)}v_y^{(1)*} - iu^{(1)*}v_y^{(1)}) \\
& - \tfrac{4}{3}\theta_z(iw^{(1)}v_y^{(1)*} - iw^{(1)*}v_y^{(1)})\Big] \\
& - 2\bar{\gamma}\bar{T}^{(1)}\Big[\bar{u}_y(u_y^{(1)*} - i\theta_x v^{(1)*}) + \bar{w}_y(w_y^{(1)*} - i\theta_z v^{(1)*}) \\
& - 2\gamma T^{(1)*}\Big[\bar{u}_y(u_y^{(1)} + i\theta_x v^{(1)}) + \bar{w}_y(w_y^{(1)} + i\theta_z v^{(1)})\Big] \\
& - \bar{\gamma}''T^{(1)}T^{(1)*}(\bar{u}_y^2 + \bar{w}_y^2) \quad , \qquad (4.28)
\end{aligned}
$$

$$k_6^{(0)} = - F_{20}\rho^{(1)}\rho^{(1)*} - F_{11}(\rho^{(1)}T^{(1)*} + \rho^{(1)*}T^{(1)})$$

$$- F_{02}T^{(1)}T^{(1)*} \quad . \tag{4.29}$$

This problem as defined by (4.22) and the appropriate boundary conditions can also be solved at each (X,Z,T') location.

At this junction $\underset{\sim}{H}^{(1)}$, $\underset{\sim}{H}^{(2)}$, and $\underset{\sim}{H}^{(0)}$ and the local dispersion relation are all known. In the problem for $\underset{\sim}{H}^{(11)}$, which determines λ, all of these functions are required. This problem is

$$(\underset{\sim}{A}_1 \frac{d^2}{dy^2} + \underset{\sim}{B}_1 \frac{d}{dy} + \underset{\sim}{C}_1)\underset{\sim}{H}^{(11)} + \underset{\sim}{K}^{(11)} + \lambda \underset{\sim}{L}^{(11)} = 0 \quad , \tag{4.30}$$

where

$$\underset{\sim}{K}^{(11)} = (k_1^{(11)}, k_2^{(11)}, k_3^{(11)}, k_4^{(11)}, k_5^{(11)}, k_6^{(11)}) \quad , \tag{4.31}$$

$$\underset{\sim}{L}^{(11)} = (\ell_1^{(11)}, \ell_2^{(11)}, \ell_3^{(11)}, \ell_4^{(11)}, \ell_5^{(11)}, \ell_6^{(11)}) \quad . \tag{4.32}$$

The functions $k_j^{(11)}$ and $\ell_j^{(11)}$ are readily calculated:

$$k_1^{(11)} = i\theta_x\left[\rho^{(2)}u^{(1)*} + \rho^{(1)*}u^{(2)} + \rho^{(0)}u^{(1)} + \rho^{(1)}u^{(0)}\right]$$

$$+ i\theta_z\left[\rho^{(2)}w^{(1)*} + \rho^{(1)*}w^{(2)} + \rho^{(0)}w^{(2)} + \rho^{(2)}w^{(0)}\right]$$

$$+ \left[\rho^{(2)}v^{(1)*} + \rho^{(1)*}v^{(2)} + \rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)}\right]_y \quad , \tag{4.33}$$

$$
\begin{aligned}
k_2^{(11)} = {} & \bar{\rho}\{\theta_x(u^{(0)}iu^{(1)} + u^{(1)*}2iu^{(2)} - iu^{(1)*}u^{(2)}) \\
& + v^{(2)}u_y^{(1)*} + v^{(1)*}u_y^{(2)} + v^{(0)}u_y^{(1)} + v^{(1)}u_y^{(0)} \\
& + \theta_z(w^{(0)}iu^{(1)} - w^{(2)}iu^{(1)*} + 2iw^{(1)*}u^{(2)})\} \\
& + \bar{\rho}\{\theta_T(\rho^{(0)}iu^{(1)} + 2i\rho^{(1)*}u^{(2)} - i\rho^{(2)}u^{(1)*}) \\
& + \bar{u}_y(\rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)} + \rho^{(2)}v^{(1)*} + \rho^{(1)*}v^{(2)}) \\
& + (\theta_x\bar{u} + \theta_z\bar{w})(i\rho^{(0)}u^{(1)} + 2i\rho^{(1)*}u^{(2)} - i\rho^{(2)}u^{(1)*})\} \\
& + \theta_x(iu^{(1)2}\rho^{(1)*}) \\
& + \theta_z(-i\rho^{(1)}w^{(1)}u^{(1)*} + i\rho^{(1)}w^{(1)*}u^{(1)} + i\rho^{(1)*}w^{(1)}u^{(1)}) \\
& + \rho^{(1)}v^{(1)}u_y^{(1)*} + \rho^{(1)}v^{(1)*}u_y^{(1)} + \rho^{(1)*}v^{(1)}u_y^{(1)} \\
& + \tfrac{2}{3}i\bar{\gamma}'\theta_x\{\theta_x(T^{(1)*}2iu^{(2)} + T^{(0)}iu^{(1)} - iT^{(2)}u^{(1)*}) \\
& \quad + \theta_z(T^{(1)*}2iw^{(2)} + T^{(0)}iw^{(1)} - iT^{(2)}w^{(1)*}) \\
& \quad + T^{(1)*}v_y^{(2)} + T^{(0)}v_y^{(1)} + T^{(1)}v_y^{(0)} + T^{(2)}v_y^{(1)*}\} \\
& - 2\bar{\gamma}'i\theta_x^2(T^{(1)*}2iu^{(2)} + T^{(0)}iu^{(1)} - T^{(2)}iu^{(1)*}) \\
& + \tfrac{1}{3}\bar{\gamma}''i\theta_x(T^{(1)2}(-i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*}) \\
& \quad + 2T^{(1)}T^{(1)*}(i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)}))
\end{aligned}
$$

$$
\begin{aligned}
&- \bar{\gamma}^{*} i\theta_x^2 (2iT^{(1)} T^{(1)*} u^{(1)} - iT^{(1)2} u^{(1)*}) \\
&- \Big[\bar{\gamma}^{*} \{ T^{(2)} (u_y^{(1)*} - i\theta_x v^{(1)*}) + T^{(0)} (u_y^{(1)} + i\theta_x v^{(1)}) \\
&\qquad + T^{(1)*} (u_y^{(2)} + 2i\theta_x v^{(2)}) \} \\
&\qquad + \bar{\gamma}^{*} \bar{u}_y \{ T^{(0)} T^{(1)} + T^{(2)} T^{(1)*} \} \Big]_y \\
&- \Big[\bar{\gamma}^{*} \{ \frac{T^{(1)2}}{2} (u_y^{(1)*} - i\theta_x v^{(1)*}) + T^{(1)} T^{(1)*} (u_y^{(1)} + i\theta_x v^{(1)}) \} \\
&\qquad + \tfrac{1}{2} \bar{\gamma}^{m} \bar{u}_y T^{(1)2} T^{(1)*} \Big]_y
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
k_3^{(11)} = \; & \bar{\rho}\{\theta_x(u^{(0)}iv^{(1)} + u^{(1)*}2iv^{(2)} - iu^{(2)}v^{(1)*}) \\
& + \theta_z(w^{(0)}iv^{(1)} + w^{(1)*}2iv^{(2)} - iw^{(2)}v^{(1)*}) \\
& + v^{(2)}v_y^{(1)*} + v^{(0)}v_y^{(1)} + v^{(1)*}v_y^{(2)}\} \\
& + \theta_T\{-i\rho^{(2)}v^{(1)*} + i\rho^{(0)}v^{(1)} + 2i\rho^{(1)*}v^{(2)}\} \\
& + \theta_x\{i(\rho^{(1)}u^{(1)*} + \rho^{(1)*}u^{(1)})v^{(1)} - i\rho^{(1)}u^{(1)}v^{(1)*}\} \\
& + \theta_z\{i(\rho^{(1)}w^{(1)*} + \rho^{(1)*}w^{(1)})v^{(1)} - i\rho^{(1)}w^{(1)}v^{(1)*}\} \\
& + \{\rho^{(1)}v^{(1)}v_y^{(1)*} + \rho^{(1)}v^{(1)*}v_y^{(1)} + \rho^{(1)*}v^{(1)}v_y^{(1)}\} \\
& + \frac{2}{3}\left[\bar{\gamma}'\{T^{(2)}(-i\theta_x u^{(1)} - i\theta_z w^{(1)} + v_y^{(1)}) \right. \\
& \qquad + T^{(0)}(i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)}) \\
& \qquad \left. + T^{(1)*}(2i\theta_x u^{(2)} + 2i\theta_z w^{(2)} + v_y^{(2)})\}\right]_y \\
& + 2\left[\bar{\gamma}'(T^{(2)}v_y^{(1)*} + T^{(0)}v_y^{(1)} + T^{(1)*}v_y^{(2)})\right]_y \\
& + \frac{1}{3}\left[\bar{\gamma}''\{2T^{(1)}T^{(1)*}(i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)}) \right. \\
& \qquad + T^{(1)2}(-i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*})\} \\
& \qquad \left. - \bar{\gamma}''\{T^{(1)2}v_y^{(1)*} + 2T^{(1)}T^{(1)*}v_y^{(1)}\}\right]_y
\end{aligned}
$$

$$
\begin{aligned}
&- i\theta_z\Big[\bar{\gamma}'T^{(2)}(w_y^{(1)*} - i\theta_z v^{(1)*}) + \bar{\gamma}'T^{(0)}(w_y^{(1)} + i\theta_z v^{(1)}) \\
&\quad + \bar{\gamma}'T^{(1)*}(w_y^{(2)} + 2i\theta_z v^{(2)}) \\
&\quad + \frac{\bar{\gamma}''}{2}\bar{w}_y(2T^{(0)}T^{(1)} + 2T^{(2)}T^{(1)*})\Big] \\
&- i\theta_z\Big[\frac{\bar{\gamma}''}{2}\{T^{(1)2}(w_y^{(1)*} - i\theta_z v^{(1)*}) \\
&\quad + 2T^{(1)}T^{(1)*}(w_y^{(1)} + i\theta_z v^{(1)})\} \\
&\quad + \frac{\gamma'''}{2}\bar{w}_y T^{(1)2}T^{(1)*}\Big] \quad , \qquad (4.35)
\end{aligned}
$$

$$
\begin{aligned}
k_4^{(11)} = \bar{\rho}\Big[&\theta_x\left(u^{(1)*}2iw^{(2)} - iu^{(2)}w^{(1)*} + iu^{(0)}w^{(1)}\right)\\
&+ v^{(2)}u_y^{(1)*} + v^{(0)}w_y^{(1)} + v^{(1)*}w_y^{(2)}\\
&+ \theta_z\left(iw^{(2)}w^{(1)*} + iw^{(0)}w^{(1)}\right)\Big]\\
&+ \theta_T\left(2i\rho^{(1)*}w^{(2)} + i\rho^{(0)}w^{(1)} - i\rho^{(2)}w^{(1)*}\right)\\
&+ \bar{w}_y\left(\rho^{(2)}v^{(1)*} + \rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)} + \rho^{(1)*}v^{(2)}\right)\\
&+ \left(\theta_x\bar{u} + \theta_z\bar{w}\right)\left(i\rho^{(0)}w^{(1)} + 2i\rho^{(1)*}w^{(2)} - i\rho^{(2)}w^{(1)*}\right)\\
&+ \theta_x\left\{-i\rho^{(1)}u^{(1)}w^{(1)*} + i\rho^{(1)}u^{(1)*}w^{(1)} + i\rho^{(1)*}u^{(1)}w^{(1)}\right\}\\
&+ \theta_z\left\{i\rho^{(1)*}w^{(1)2}\right\}\\
&+ \left\{\rho^{(1)}v^{(1)}w_y^{(1)*} + \rho^{(1)}v^{(1)*}w_y^{(1)} + \rho^{(1)*}v^{(1)}w_y^{(1)}\right\}\\
&- i\theta_x\bar{\gamma}'\left\{\theta_z\left(iT^{(0)}u^{(1)} - iT^{(2)}u^{(1)*} + 2iT^{(1)*}u^{(2)}\right)\right.\\
&\qquad\left. + \theta_x\left(iT^{(0)}w^{(1)} - iT^{(2)}w^{(1)*} + 2iT^{(1)*}w^{(2)}\right)\right\}\\
&- i\theta_x\bar{\gamma}''\left\{\theta_z\left(-iT^{(1)2}u^{(1)*} + 2iT^{(1)}T^{(1)*}u^{(1)}\right)\right.\\
&\qquad\left. + \theta_x\left(-iT^{(1)2}w^{(1)*} + 2iT^{(1)}T^{(1)*}w^{(1)}\right)\right\}\\
&- \Big[\bar{\gamma}'\left\{T^{(2)}\left(w_y^{(1)*} - i\theta_z v^{(1)*}\right) + T^{(0)}\left(w_y^{(1)} + i\theta_z v^{(1)}\right)\right.\\
&\qquad\left. + T^{(1)*}\left(w_y^{(2)} - 2i\theta_z v^{(2)}\right)\right\}\\
&\quad + \bar{\gamma}''\bar{w}_y\left\{T^{(0)}T^{(1)} + T^{(2)}T^{(1)*}\right\}\Big]_y
\end{aligned}
$$

$$
- \left[\frac{\bar{\gamma}''}{2}\{T^{(1)^2}(w_y^{(1)*} - i\theta_z v^{(1)*}) + 2T^{(1)}T^{(1)*}(w_y^{(1)} + i\theta_z v^{(1)})\}\right.
$$

$$
\left. + \frac{\bar{\gamma}'''}{2}\bar{w}_y\{T^{(1)^2}T^{(1)*}\}\right]_y
$$

$$
+ i\theta_z\left[\frac{2}{3}\bar{\gamma}'\{\theta_x(-iT^{(2)}u^{(1)*} + iT^{(0)}u^{(1)} + 2iT^{(1)*}u^{(2)})\right.
$$

$$
+ \theta_z(-iT^{(2)}w^{(1)*} + iT^{(0)}w^{(1)} + 2iT^{(1)*}w^{(2)})
$$

$$
\left. + (T^{(2)}v_y^{(1)*} + T^{(0)}v_y^{(1)} + T^{(1)*}v_y^{(2)})\}\right.
$$

$$
\left. - 2\bar{\gamma}'\theta_z(-iT^{(2)}w^{(1)*} + iT^{(0)}w^{(1)} + 2iT^{(1)*}w^{(2)})\right]
$$

$$
+ i\theta_z\left[\frac{1}{3}\bar{\gamma}''\{\theta_x(-iT^{(1)^2}u^{(1)*} + 2iT^{(1)}T^{(1)*}u^{(1)})\right.
$$

$$
+ \theta_z(-iT^{(1)^2}w^{(1)*} + 2iT^{(1)}T^{(1)*}w^{(1)})
$$

$$
\left. + (T^{(1)^2}v_y^{(1)*} + 2T^{(1)}T^{(1)*}v_y^{(1)})\}\right]
$$

$$
- i\theta_z^2\gamma''\left[-iT^{(1)^2}w^{(1)*} + 2iT^{(1)}T^{(1)*}w^{(1)}\right] , \qquad (4.36)
$$

$$
\begin{aligned}
k_5^{(11)} = \bar{\rho}\bar{c}_v\Big[&\theta_x\{-iu^{(2)}T^{(1)*} + iu^{(0)}T^{(1)} + 2iu^{(1)*}T^{(2)}\} \\
&+ \theta_z\{-iw^{(2)}T^{(1)*} + iw^{(0)}T^{(1)} + 2iw^{(1)*}T^{(2)}\} \\
&+ \{v^{(2)}T_y^{(1)*} + v^{(1)*}T_y^{(2)} + v^{(0)}T_y^{(1)} + v^{(1)}T_y^{(0)}\}\Big] \\
&+ \bar{c}_v(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z)(2i\rho^{(1)*}T^{(2)} + i\rho^{(0)}T^{(1)} - i\rho^{(2)}T^{(1)*}) \\
&+ \bar{c}_v\bar{T}_y(\rho^{(1)*}v^{(2)} + \rho^{(0)}v^{(1)} + \rho^{(1)}v^{(0)} + \rho^{(2)}v^{(1)*}) \\
&+ i\bar{\rho}\bar{c}_v'(\theta_T + \bar{u}\theta_x + \bar{w}\theta_z)(2T^{(0)}T^{(1)} + 2T^{(2)}T^{(1)*}) \\
&+ \bar{\rho}\bar{c}_v'\bar{T}_y(T^{(1)*}v^{(2)} + T^{(0)}v^{(1)} + T^{(1)}v^{(0)} + T^{(2)}v^{(1)*}) \\
&+ i\bar{c}_v\theta_x(\rho^{(1)*}u^{(1)}T^{(1)} + \rho^{(1)}u^{(1)*}T^{(1)} - \rho^{(1)}u^{(1)}T^{(1)*}) \\
&+ i\bar{c}_v\theta_z(\rho^{(1)*}w^{(1)}T^{(1)} + \rho^{(1)}w^{(1)*}T^{(1)} - \rho^{(1)}w^{(1)}T^{(1)*}) \\
&+ \bar{c}_v(\rho^{(1)}v^{(1)}T_y^{(1)*} + \rho^{(1)}v^{(1)*}T_y^{(1)} + \rho^{(1)*}v^{(1)}T_y^{(1)}) \\
&+ i\bar{\rho}\bar{c}_v'\theta_x(2u^{(1)*}T^{(1)^2}) + i\bar{\rho}\bar{c}_v'\theta_z(2w^{(1)*}T^{(1)^2}) \\
&+ \bar{\rho}\bar{c}_v'(v^{(1)*}T^{(1)}T_y^{(1)} + v^{(1)}(T^{(1)}T^{(1)*})_y) \\
&+ \bar{c}_v(\theta_T + \theta_x\bar{u} + \theta_z\bar{w})iT^{(1)^2}\rho^{(1)*} \\
&+ \bar{c}_v\bar{T}_y(\rho^{(1)}T^{(1)}v^{(1)*} + \rho^{(1)}T^{(1)*}v^{(1)} + \rho^{(1)*}T^{(1)}v^{(1)})
\end{aligned}
$$

$$\frac{1}{2}\bar{\rho}\bar{c}_v^*(\theta_T + \theta_x u + \theta_z w) i T^{(1)^2} T^{(1)*}$$

$$+ \frac{1}{2}\bar{\rho}\bar{c}_v^*\bar{T}_y (T^{(1)^2} v^{(1)*} + 2T^{(1)} T^{(1)*} v^{(1)})$$

$$- \bar{\gamma}\Big[2u_y^{(0)} u_y^{(1)} + 2u_y^{(2)} u_y^{(1)*}$$

$$+ 2i\theta_x (u_y^{(0)} v^{(1)} - u_y^{(1)} v^{(0)} + 2u_y^{(1)*} v^{(2)} - u_y^{(2)} v^{(1)*})$$

$$+ 2w_y^{(0)} w_y^{(1)} + 2w_y^{(2)} w_y^{(1)*}$$

$$+ 2i\theta_z (w_y^{(0)} v^{(1)} + w_y^{(1)} v^{(0)} + 2w_y^{(1)*} v^{(2)} - w_y^{(2)} v^{(1)*})$$

$$+ 4(\theta_x^2 + \theta_z^2) v^{(2)} v^{(1)*} + 4\theta_z^2 u^{(2)} u^{(1)*}$$

$$+ 4\theta_x^2 w^{(2)} w^{(1)*} - 4\theta_x\theta_z (u^{(1)*} w^{(2)} + u^{(2)} w^{(1)*})\Big]$$

$$- \frac{4}{3}\bar{\gamma}\Big[4\theta_x^2 u^{(2)} u^{(1)*} + 4\theta_z^2 w^{(2)} w^{(1)*} + 2(v_y^{(2)} v_y^{(1)*} + v_y^{(0)} v_y^{(1)})$$

$$- i\theta_x (v_y^{(0)} u^{(1)} - v_y^{(2)} u^{(1)*} + 2v_y^{(1)*} u^{(2)})$$

$$- i\theta_z (v_y^{(0)} w^{(1)} - v_y^{(2)} w^{(1)*} + 2v_y^{(1)*} w^{(2)})$$

$$+ 2\theta_x\theta_z (u^{(2)} w^{(1)*} + u^{(1)*} w^{(2)})\Big]$$

$$- 2\bar{\gamma}'\bar{u}_y\Big[T^{(2)} (u_y^{(1)*} - i\theta_x v^{(1)*}) + T^{(0)} (u_y^{(1)} + i\theta_x v^{(1)})$$

$$+ T^{(1)*} (u_y^{(2)} + 2i\theta_x v^{(2)})\Big]$$

$$- 2\bar{\gamma}'\bar{w}_y\Big[T^{(2)}(w_y^{(1)*} - i\theta_z v^{(1)*}) + T^{(0)}(w_y^{(1)} + i\theta_z v^{(1)})$$

$$+ T^{(1)*}(w_y^{(2)} + 2i\theta_z v^{(2)})\Big]$$

$$- \bar{\gamma}''(\bar{u}_y^2 + \bar{w}_y^2)(T^{(0)}T^{(1)} + T^{(2)}T^{(1)*})$$

$$- \gamma'\Big[T^{(1)}\{2(u_y^{(1)} + i\theta_x v^{(1)})(u_y^{(1)*} - i\theta_x v^{(1)*})$$

$$+ 2(w_y^{(1)} + i\theta_z v^{(1)})(w_y^{(1)*} - i\theta_z v^{(1)*})$$

$$+ 2(\theta_x u^{(1)} + \theta_z w^{(1)})(\theta_x u^{(1)*} + \theta_z w^{(1)*})\}$$

$$+ T^{(1)*}\{(u_y^{(1)} + i\theta_x v^{(1)})^2 + (w_y^{(1)} + i\theta_z v^{(1)})^2$$

$$- (\theta_z u^{(1)} + \theta_x w^{(1)})^2\}\Big]$$

$$- \frac{4}{3}\gamma'\Big[\theta_x^2\{T^{(1)}2u^{(1)}u^{(1)*} - T^{(1)*}u^{(1)2}\}$$

$$+ \{T^{(1)}2v_y^{(1)}v_y^{(1)*} + T^{(1)*}v_y^{(1)2}\}$$

$$+ \theta_z^2\{T^{(1)}2w^{(1)}w^{(1)*} - T^{(1)*}w^{(1)2}\}$$

$$- i\theta_x(T^{(1)}u^{(1)}v_y^{(1)*} + T^{(1)*}u^{(1)}v_y^{(1)} - T^{(1)}u^{(1)*}v_y^{(1)})$$

$$- i\theta_z(T^{(1)}w^{(1)}v_y^{(1)*} + T^{(1)*}w^{(1)}v_y^{(1)} - T^{(1)}w^{(1)*}v_y^{(1)})$$

$$+ \theta_x\theta_z(T^{(1)*}u^{(1)}w^{(1)} - T^{(1)}u^{(1)}w^{(1)*} - T^{(1)}u^{(1)*}w^{(1)})\Big]$$

$$- \bar{\gamma}^* \bar{u}_y \left[T^{(1)^2} (u_y^{(1)*} - i\theta_x v^{(1)*}) + 2T^{(1)} T^{(1)*} (u_y^{(1)} + i\theta_x v^{(1)}) \right]$$

$$- \bar{\gamma}^* \bar{w}_y \left[T^{(1)^2} (w_y^{(1)*} - i\theta_z v^{(1)*}) + 2T^{(1)} T^{(1)*} (w_y^{(1)} + i\theta_z v^{(1)}) \right]$$

$$- \frac{\bar{\gamma}^{*'}}{2} (\bar{u}_y^2 + \bar{w}_y^2) T^{(1)^2} T^{(1)*}$$

$$+ \bar{T} F_{11} \left[\rho^{(2)} (-i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*}) \right.$$

$$+ \rho^{(0)} (i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)})$$

$$\left. + \rho^{(1)*} (2i\theta_x u^{(2)} + 2i\theta_z w^{(2)} + v_y^{(2)}) \right]$$

$$+ \{ T F_{02} + F_{01} \} \left[T^{(2)} (-i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*}) \right.$$

$$+ T^{(0)} (i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)})$$

$$\left. + T^{(1)*} (2i\theta_x u^{(2)} + 2i\theta_z w^{(2)} + v_y^{(2)}) \right]$$

$$+ \bar{T} \{ \tfrac{1}{2} F_{21} \rho^{(1)^2} + F_{12} \rho^{(1)} T^{(1)} + \tfrac{1}{2} F_{03} T^{(1)^2} \}$$

$$\{ -i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*} \}$$

$$+ \bar{T} \{ F_{21} \rho^{(1)} \rho^{(1)*} + F_{12} (\rho^{(1)} T^{(1)*} + \rho^{(1)*} T^{(1)}) + F_{03} T^{(1)} T^{(1)*} \}$$

$$\{ i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)} \}$$

$$+ \{F_{11}\rho^{(1)}T^{(1)} + F_{02}T^{(1)^2}\}\{-i\theta_x u^{(1)*} - i\theta_z w^{(1)*} + v_y^{(1)*}\}$$

$$+ \{F_{11}(\rho^{(1)}T^{(1)*} + \rho^{(1)*}T^{(1)}) + 2F_{02}T^{(1)}T^{(1)*}\}$$

$$\{ i\theta_x u^{(1)} + i\theta_z w^{(1)} + v_y^{(1)}\}$$

$$+ \bar{k}'(\theta_x^2 + \theta_z^2)(T^{(0)}T^{(1)} + T^{(2)}T^{(1)*})$$

$$- \bar{k}'(T^{(0)}T^{(1)} + T^{(2)}T^{(1)*})_{yy}$$

$$+ \tfrac{1}{2}\bar{k}''(\theta_x^2 + \theta_z^2)T^{(1)^2}T^{(1)*}$$

$$- \tfrac{1}{2}(\bar{k}''T^{(1)^2}T^{(1)*})_{yy} \quad , \tag{4.37}$$

$$k_6^{(11)} = - F_{20}(\rho^{(0)}\rho^{(1)} + \rho^{(2)}\rho^{(1)*})$$

$$- F_{11}(\rho^{(0)}T^{(1)} + \rho^{(1)}T^{(0)} + \rho^{(2)}T^{(1)*} + \rho^{(1)*}T^{(2)})$$

$$- F_{02}(T^{(0)}T^{(1)} + T^{(2)}T^{(1)*})$$

$$- \tfrac{1}{2}F_{30}\rho^{(1)^2}\rho^{(1)*} - \tfrac{1}{2}F_{21}(\rho^{(1)^2}T^{(1)*} + 2\rho^{(1)}\rho^{(1)*}T^{(1)})$$

$$- \tfrac{1}{2}F_{12}(T^{(1)^2}\rho^{(1)*} + 2T^{(1)}T^{(1)*}\rho^{(1)}) - \tfrac{1}{2}F_{03}T^{(1)^2}T^{(1)*} \quad ,$$

(4.38)

$$\ell_1^{(11)} = \rho^{(1)}, \quad \ell_2^{(11)} = \bar{\rho}u^{(1)}, \quad \ell_3^{(11)} = \bar{\rho}v^{(1)}, \tag{4.39}$$

$$\ell_4^{(11)} = \bar{\rho}w^{(1)}, \quad \ell_5^{(11)} = \bar{\rho}\bar{c}_v T^{(1)}, \quad \ell_6^{(11)} = 0 . \tag{4.40}$$

For the nonparallel contributions we must examine the series of problems

$$(A_{\ell 1}\frac{d^2}{dy^2} + B_{\ell 1}\frac{d}{dy} + C_{\ell 1})H_j^{(1)} + K_j^{(1)} + \alpha_j L^{(1)} = 0 ,$$

$$j = 1,2,3, \tag{4.41}$$

where

$$K_j^{(1)} = (k_{j1}^{(1)}, k_{j2}^{(1)}, k_{j3}^{(1)}, k_{j4}^{(1)}, k_{j5}^{(1)}, k_{j6}^{(1)}) , \tag{4.42}$$

$$L^{(1)} = (\ell_1^{(1)}, \ell_2^{(1)}, \ell_3^{(1)}, \ell_4^{(1)}, \ell_5^{(1)}, \ell_6^{(1)}) . \tag{4.43}$$

These functions, twenty-four of them in all, are now listed.

$$k_{11}^{(1)} = \rho^{(1)}\bar{u} + \bar{\rho}u^{(1)} , \tag{4.44}$$

$$k_{12}^{(1)} = \bar{\rho}\bar{u}v^{(1)} + \rho^{(1)} + \frac{2}{3}\bar{\gamma}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$- 2\bar{\gamma}i\theta_x u^{(1)} + \frac{2}{3}\bar{\gamma}iu^{(1)}\theta_x - 2\bar{\gamma}i\theta_x u^{(1)}$$

$$- (\bar{\gamma}v^{(1)})_y - \bar{\gamma}i\theta_z w^{(1)} , \tag{4.45}$$

$$k_{13}^{(1)} = \bar{\rho}\bar{u}v^{(1)} - \bar{\gamma}\theta_x iv^{(1)} - \bar{\gamma}(u_y^{(1)} + i\theta_x v^{(1)})$$

$$- \bar{\gamma}'\bar{u}_y T^{(1)} + \frac{2}{3}(\bar{\gamma}u^{(1)})_y \quad , \tag{4.46}$$

$$k_{14}^{(1)} = \bar{\rho}\bar{u}w^{(1)} - \bar{\gamma}(i\theta_z u^{(1)} + i\theta_x w^{(1)})$$

$$- \bar{\gamma}i\theta_x w^{(1)} + \frac{2}{3}\bar{\gamma}\theta_z iu^{(1)} \quad , \tag{4.47}$$

$$k_{15}^{(1)} = \bar{\rho}\bar{c}_v\bar{u}T^{(1)} - 2\bar{\gamma}\bar{u}_y v^{(1)} + \bar{T}F_{01}u^{(1)}$$

$$- i\bar{k}\theta_x T^{(1)} - i\theta_x \bar{k}T^{(1)} \quad , \tag{4.48}$$

$$k_{16}^{(1)} = 0 \quad , \tag{4.49}$$

$$\ell_1^{(1)} = \rho^{(1)}, \quad \ell_2^{(1)} = \bar{\rho}u^{(1)}, \quad \ell_3^{(1)} = \bar{\rho}v^{(1)} \quad , \tag{4.50}$$

$$\ell_4^{(1)} = \bar{\rho}w^{(1)}, \quad \ell_5^{(1)} = \bar{\rho}\bar{c}_v T^{(1)}, \quad \ell_6^{(1)} = 0 \quad , \tag{4.51}$$

$$k_{21}^{(1)} = \rho^{(1)}\bar{w} + \bar{\rho}w^{(1)} \quad , \tag{4.52}$$

$$k_{22}^{(1)} = \bar{\rho}\bar{w}u^{(1)} + \frac{2}{3}\bar{\gamma}\theta_x iw^{(1)} - \bar{\gamma}\theta_z iu^{(1)}$$

$$- i(\theta_z u^{(1)} + \theta_x w^{(1)}) \quad , \tag{4.53}$$

$$k_{23}^{(1)} = \bar{\rho}\bar{w}v^{(1)} + \frac{2}{3}(\bar{\gamma}w^{(1)})_y - i\bar{\gamma}\theta_z v^{(1)}$$

$$- \bar{\gamma}(w_y^{(1)} + i\theta_z v^{(1)}) - \bar{\gamma}'\bar{w}_y T^{(1)} \quad , \tag{4.54}$$

$$k_{24}^{(1)} = \bar{\rho}\bar{w}w^{(1)} - \bar{\gamma}i\theta_x u^{(1)} + \rho^{(1)}$$

$$+ \frac{2}{3}\bar{\gamma}\left[i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)}\right]$$

$$- 2\bar{\gamma}\theta_z i w^{(1)} - (\bar{\gamma}v^{(1)})_y$$

$$+ \frac{2}{3}\bar{\gamma}i\theta_z w^{(1)} - 2\bar{\gamma}\theta_z i w^{(1)} \quad , \tag{4.55}$$

$$k_{25}^{(1)} = \bar{\rho}\bar{c}_v\bar{w}T^{(1)} - 2\bar{\gamma}\bar{w}_y v^{(1)} + \bar{T}F_{01}w^{(1)}$$

$$- \bar{k}\theta_z i T^{(1)} - \theta_z \bar{k} i T^{(1)} \quad , \tag{4.56}$$

$$k_{26}^{(1)} = 0 \quad . \tag{4.57}$$

$$k_{31}^{(1)} = (\bar{u}_x + \bar{v}_y + \bar{w}_z)\rho^{(1)} + \bar{\rho}_x u^{(1)} + \bar{\rho}_y v^{(1)} + \bar{\rho}_z w^{(1)}$$

$$+ i(\bar{\bar{u}}\theta_x + \bar{\bar{w}}\theta_z)\rho^{(1)} + i\bar{\bar{\rho}}(\theta_x u^{(1)} + \theta_z w^{(1)})$$

$$+ (\bar{\bar{\rho}}v^{(1)})_y + \rho_x^{(1)}\bar{u} + \bar{\rho}u_x^{(1)}$$

$$+ \rho_z^{(1)}\bar{w} + \bar{\rho}w_z^{(1)} \quad , \tag{4.58}$$

$$
\begin{aligned}
k_{32}^{(1)} = {} & \overline{\rho v} u_y^{(1)} + \overline{\rho u}_x u^{(1)} + \overline{\rho u}_z w^{(1)} \\
& + (\bar{u}_{T'} + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z)\rho^{(1)} \\
& + \tfrac{2}{3}\gamma(i\theta_{xx}u^{(1)} + i\theta_{xz}w^{(1)}) \\
& + \tfrac{2}{3}\bar{\gamma}'\bar{T}_x(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)}) \\
& - 2\bar{\gamma}\theta_{xx} i u^{(1)} - 2\bar{\gamma}'\bar{T}_x i\theta_x u^{(1)} \\
& + \tfrac{2}{3}\bar{\gamma}'\theta_x iT^{(1)}(\bar{u}_x + \bar{v}_y + \bar{w}_z) \\
& - 2\bar{\gamma}'\theta_x\bar{u}_x iT^{(1)} - i\bar{\gamma}'\theta_z T^{(1)}(\bar{u}_z + \bar{w}_x) \\
& - \bar{\gamma}(\theta_{zz} i u^{(1)} + i\theta_{xz} w^{(1)}) \\
& - \bar{\gamma}'\bar{T}_z(i\theta_z u^{(1)} + i\theta_x w^{(1)}) \\
& + i\{(\overline{\bar{\rho}\bar{u}} + \overline{\bar{\rho}\bar{u}})\theta_x + (\overline{\bar{\rho}\bar{w}} + \overline{\bar{\rho}\bar{w}})\theta_z\}u^{(1)} \\
& + (\overline{\bar{\rho}\bar{u}}_y + \overline{\bar{\rho}\bar{u}}_y)v^{(1)} \\
& - \tfrac{2}{3}\bar{\gamma}'\bar{\bar{T}}\theta_x(\theta_x u^{(1)} + \theta_z w^{(1)} - i v_y^{(1)}) + 2\bar{\gamma}'\bar{\bar{T}}\theta_x^2 u^{(1)} \\
& + \bar{\gamma}'\bar{\bar{T}}\theta_z(\theta_z u^{(1)} + \theta_x w^{(1)}) - \{\bar{\gamma}'\bar{\bar{T}}(u_y^{(1)} + i\theta_x v^{(1)})\}_y \\
& - ((\bar{\gamma}'\bar{\bar{u}}_y + \bar{\gamma}''\bar{u}_y\bar{\bar{T}})T^{(1)})_y + \overline{\bar{\rho}\bar{u}} u_x^{(1)} + \rho_x^{(1)}
\end{aligned}
$$

$$+ \frac{2}{3}\bar{\gamma}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$- \frac{10}{3} i\bar{\gamma}\theta_x u^{(1)} - i\bar{\gamma}\theta_z w^{(1)} - (\bar{\gamma}v^{(1)})_y$$

$$+ \bar{\rho}\bar{w}u_z^{(1)} - \frac{1}{3} i\bar{\gamma}\theta_x w_z^{(1)} , \qquad (4.59)$$

$$k_{33}^{(1)} = \overline{\rho v}_y v^{(1)} - \gamma\theta_{xx} iv^{(1)} - \bar{\gamma}'\bar{T}_x(u_y^{(1)} + i\theta_x v^{(1)})$$

$$- (\bar{\gamma}'\bar{u}_y)_x T^{(1)} + \frac{2}{3}(\bar{\gamma}'(\bar{u}_x + \bar{v}_y + \bar{w}_z)T^{(1)})_y$$

$$- 2(\bar{\gamma}'\bar{v}_y T^{(1)})_y - \bar{\gamma}'\bar{T}_z(w_y^{(1)} + i\theta_z v^{(1)})$$

$$- \bar{\gamma}\theta_{zz} iv^{(1)} - (\bar{\gamma}'\bar{w}_y)_z T^{(1)}$$

$$+ i\{(\bar{\rho}\bar{\bar{u}} + \bar{\bar{\rho}}\bar{u})\theta_x + (\bar{\rho}\bar{\bar{w}} + \bar{\bar{\rho}}\bar{w})\theta_z\}v^{(1)}$$

$$- i\bar{\gamma}'\bar{\bar{T}}(u_y^{(1)} + i\theta_x v^{(1)})$$

$$- i\theta_x(\bar{\gamma}'\bar{\bar{u}}_y + \bar{\gamma}'\bar{u}_y\bar{\bar{T}})T^{(1)} - i\theta_z(\bar{\gamma}'\bar{\bar{w}}_y + \bar{\gamma}'\bar{w}_y\bar{\bar{T}})T^{(1)}$$

$$- i\bar{\gamma}'\bar{\bar{T}}\theta_z(w_y^{(1)} + i\theta_z v^{(1)}) - 2(\bar{\gamma}'\bar{\bar{T}}v_y^{(1)})_y$$

$$- \frac{2}{3}\bar{\gamma}'\bar{\bar{T}}(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$+ \bar{\rho}\bar{u}v_x^{(1)} - i\theta_x\bar{\gamma}v_x^{(1)} - \bar{\gamma}(u_{yx}^{(1)} + i\theta_x v_x^{(1)}) - \bar{\gamma}\bar{u}_y T_x^{(1)}$$

$$+ \frac{2}{3}(\bar{\gamma}u_x^{(1)})_y + \bar{\rho}\bar{w}v_z^{(1)} + \frac{2}{3}(\bar{\gamma}w_z^{(1)})_y - i\gamma\theta_z v_z^{(1)}$$

$$- \bar{\gamma}(w_{yz}^{(1)} + i\theta_z v_z^{(1)}) - \bar{\gamma}\bar{w}_y T_z^{(1)} \quad , \tag{4.60}$$

$$k_{34}^{(1)} = \bar{\rho}\bar{v}w_y^{(1)} + \bar{\rho}\bar{w}_z w^{(1)} + \bar{\rho}\bar{w}_x u^{(1)} + \rho^{(1)}(\bar{w}_T + \bar{u}\bar{w}_x + \bar{v}\bar{w}_y + \bar{w}\bar{w}_z)$$

$$- \bar{\gamma}'\bar{T}_x(i\theta_z u^{(1)} + i\theta_x w^{(1)}) - \bar{\gamma}(\theta_{zx} iu^{(1)} + i\theta_{xx} w^{(1)})$$

$$- \bar{\gamma}' i\theta_x T^{(1)}(\bar{u}_z + \bar{w}_x) + \frac{2}{3}\bar{\gamma}'\bar{T}_z(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$+ \frac{2}{3}\bar{\gamma}(i\theta_{xz} u^{(1)} + i\theta_{zz} w^{(1)}) - 2(\bar{\gamma}\theta_z)_z iw^{(1)}$$

$$+ \frac{2}{3}\bar{\gamma}\theta_z iT^{(1)}(\bar{u}_x + \bar{v}_y + \bar{w}_z) - 2\bar{\gamma}'\theta_z\bar{w}_z iT^{(1)}$$

$$+ \{(\bar{\rho}\bar{\bar{u}} + \bar{\bar{\rho}}\bar{u})\theta_x + (\bar{\rho}\bar{\bar{w}} + \bar{\bar{\rho}}\bar{w})\theta_z\}w^{(1)} + (\bar{\rho}\bar{\bar{w}}_y + \bar{\bar{\rho}}\bar{w}_y)v^{(1)}$$

$$- \frac{2}{3}\bar{\gamma}'\bar{\bar{T}}\theta_z(\theta_x u^{(1)} + \theta_z w^{(1)} - iv_y^{(1)})$$

$$- \{\bar{\gamma}'\bar{\bar{T}}(w_y^{(1)} + i\theta_z v^{(1)})\}_y$$

$$+ 2\bar{\gamma}'\bar{\bar{T}}\theta_z^2 w^{(1)} - ((\bar{\gamma}'\bar{\bar{w}}_y + \bar{\gamma}'\bar{\bar{T}}w_y)T^{(1)})_y$$

$$- \{\bar{\gamma}'\bar{\bar{T}}(w_y^{(1)} + i\theta_z v^{(1)})\}_y + \bar{\gamma}'\bar{\bar{T}}\theta_x(\theta_x u^{(1)} + \theta_z w^{(1)})$$

$$+ \bar{\rho}\bar{u}w_x^{(1)} - \bar{\gamma}(i\theta_z u_x^{(1)} + i\theta_x w_x^{(1)}) - i\bar{\gamma}\theta_x w_x^{(1)}$$

$$+ \frac{2}{3} i\bar{\gamma}\theta_z u_x^{(1)} + \overline{\rho w} w_z^{(1)} - i\theta_x \bar{\gamma} u_z^{(1)} + \rho_z^{(1)} - 4i\theta_z \bar{\gamma} w_z^{(1)}$$

$$+ \frac{2}{3}\bar{\gamma}\left[i\theta_x u_z^{(1)} + v_{yz}^{(1)} + i\theta_z w_z^{(1)}\right] - (\bar{\gamma} v^{(1)})_{yz}$$

$$+ \frac{2}{3} i\bar{\gamma}\theta_z w_z^{(1)} \quad . \tag{4.61}$$

$$k_{35}^{(1)} = \bar{\rho}\bar{c}_v(\bar{\gamma} T_y^{(1)} + \bar{T}_x u^{(1)} + \bar{T}_z w^{(1)})$$

$$- \gamma\Big[2i(\bar{u}_z + \bar{w}_x)(\theta_z u^{(1)} + \theta_x w^{(1)})$$

$$+ \frac{4}{3}(2i\bar{u}_x\theta_x u^{(1)} + 2\bar{v}_y v_y^{(1)} + 2i\bar{w}_z\theta_z w^{(1)}$$

$$- (\bar{u}_x v_y^{(1)} + i\theta_x \bar{v}_y u^{(1)}) - (\bar{w}_z v_y^{(1)} + i\theta_z \bar{v}_y w^{(1)})\}\Big]$$

$$+ \bar{T}(\bar{u}_x + \bar{v}_y + \bar{w}_z)(F_{11}\rho^{(1)} + F_{02}T^{(1)})$$

$$+ F_{01}(\bar{u}_x + \bar{v}_y + \bar{w}_z)T^{(1)}$$

$$- i\bar{k}'(\bar{T}_x\theta_x + \bar{T}_z\theta_z)T^{(1)} - i\{(\bar{k}\theta_x)_x + (\bar{k}\theta_z)_z\}T^{(1)}$$

$$+ (\bar{\bar{\rho}}\bar{c}_v + \bar{\rho}\bar{c}_v'\bar{\bar{T}})\theta_T iT^{(1)} + (\bar{\bar{\rho}}\bar{c}_v\bar{u} + \bar{\rho}\bar{c}_v'\bar{\bar{T}}u + \bar{\rho}\bar{c}_v\bar{\bar{u}})i\theta_x T^{(1)}$$

$$+ (\bar{\rho}\bar{c}_v\bar{\bar{T}}_y + \bar{\rho}\bar{c}_v'\bar{\bar{T}}\bar{T}_y + \bar{\bar{\rho}}\bar{c}_v\bar{T}_y)v^{(1)}$$

$$- (2\bar{\gamma}\bar{\bar{u}}_y + 2\bar{\gamma}'\bar{\bar{T}}\bar{u}_y)(u_y^{(1)} + i\theta_x v^{(1)})$$

$$- (2\bar{\gamma}\bar{\bar{w}}_y + 2\bar{\gamma}'\bar{\bar{T}}\bar{w}_y)(w_y^{(1)} + i\theta_z v^{(1)})$$

$$- 2\bar{\gamma}'(\bar{u}_y\bar{\bar{u}}_y + \bar{w}_y\bar{\bar{w}}_y)T^{(1)} - \bar{\gamma}'\bar{\bar{T}}(\bar{u}_y^2 + \bar{w}_y^2)T^{(1)}$$

$$- \bar{\gamma}'(\bar{u}_y\bar{\bar{u}}_y + \bar{w}_y\bar{\bar{w}}_y)T^{(1)} - \bar{\gamma}'\bar{\bar{T}}(\bar{u}_y^2 + \bar{w}_y^2)T^{(1)}$$

$$+ (\bar{\bar{T}}F_{01} + \bar{T}(F_{11}\bar{\bar{\rho}} + F_{02}\bar{\bar{T}}))(i\theta_x u^{(1)} + v_y^{(1)} + i\theta_z w^{(1)})$$

$$+ \bar{k}'\bar{\bar{T}}(\theta_x^2 + \theta_z^2)T^{(1)} - (\bar{k}'\bar{\bar{T}}T_y^{(1)})_y$$

$$+ \bar{\rho}\bar{c}_v\bar{u}T_x^{(1)} - 2\bar{\gamma}\bar{\bar{u}}_y v_x^{(1)} - \bar{T}F_{01}u_x^{(1)} - 2i\bar{k}\theta_x T_x^{(1)}$$

$$+ \bar{\rho}\bar{c}_v\bar{w}T_z^{(1)} - 2\bar{\gamma}\bar{\bar{w}}_y v_z^{(1)} - \bar{T}F_{01}w_z^{(1)} - 2i\bar{k}\theta_z T_z^{(1)} \quad . \qquad (4.62)$$

$$k_{36}^{(1)} = -(F_{20}\bar{\bar{\rho}} + F_{11}\bar{\bar{T}})\rho^{(1)} - (F_{11}\bar{\bar{\rho}} + F_{02}\bar{\bar{T}})T^{(1)} \quad . \qquad (4.63)$$

The double bar symbols in these $k_{3j}^{(1)}$ equations correspond to the first correction to the basic boundary layer flow. This completes the detailed listings of these functions.

It remains to show how λ and α_j $j = 1,2,3$ are to be calculated. For this purpose the return to the basic linear eigenvalue problem associated with equations (4.3,4,5,6,7,8). This can be written as

$$L\underset{\sim}{\chi} = \underset{\sim}{D}_0 \frac{d^2\underset{\sim}{\chi}}{dy^2} + \underset{\sim}{D}_1 \frac{d\underset{\sim}{\chi}}{dy} + \underset{\sim}{D}_2\underset{\sim}{\chi} = 0 \quad . \qquad (4.64)$$

subject to the appropriate boundary conditions. Here $\underset{\sim}{\chi}$ is the vector $(\rho^{(1)}, u^{(1)}, v^{(1)}, w^{(1)}, T^{(1)}, p^{(1)})$ and $\underset{\sim}{D}_j$ $j = 0,1,2$ are 6×6 matrices.

The adjoint problem is that associated with the equation

$$\underset{\sim}{M}\underset{\sim}{w} = \frac{d^2}{dy^2}(\underset{\sim}{D}_0^T\underset{\sim}{w}) - \frac{d}{dy}(\underset{\sim}{D}_1^T\underset{\sim}{w}) + \underset{\sim}{D}_2^T\underset{\sim}{w} = 0 \quad . \tag{4.65}$$

and the adjoint boundary conditions follow the identity

$$\int_0^\infty (\underset{\sim}{w}^T\underset{\sim}{L}\underset{\sim}{v} - \underset{\sim}{v}^T\underset{\sim}{M}\underset{\sim}{w})dy = \underset{\sim}{w}^T\underset{\sim}{D}_0\frac{d\underset{\sim}{v}}{dy} + (\underset{\sim}{w}^T\underset{\sim}{D}_1 - \frac{d}{dy}(\underset{\sim}{w}^T\underset{\sim}{D}_0))\underset{\sim}{v}\Big]_0^\infty \tag{4.66}$$

It follows that the inhomogeneous problem

$$\underset{\sim}{L}\underset{\sim}{v} = \underset{\sim}{f} \tag{4.67}$$

will have a solution if and only if

$$\int_0^\infty \underset{\sim}{w}^T\underset{\sim}{f}dy = 0 \quad . \tag{4.68}$$

On applying this condition to equations (4.30) and (4.41) we obtain

$$\lambda = -\frac{\int_0^\infty \underset{\sim}{w}^T\underset{\sim}{K}^{(11)}dy}{\int_0^\infty \underset{\sim}{w}^T\underset{\sim}{L}^{(11)}dy} \tag{4.69}$$

$$\alpha_j = -\frac{\int_0^\infty w^T K_j^{(1)}dy}{\int_0^\infty w^T L^{(1)}dy} \quad , \qquad j = 1,2,3 \tag{4.70}$$

With these functions determined at each location the coefficients in equation governing the amplitude A (equation 4.12) are known and the spatial and temporal growth of A can be studied.

5. CHEBYSHEV-SPECTRAL METHODS FOR STABILITY CALCULATIONS

In Sects. 5-8, we discuss the state-of-the-art of numerical techniques for stability calculations. In this Section, we provide an introduction to the use of Chebyshev polynomials for the solution of stability problems.[9]

An important difference between finite-difference approximations to the eigenfunctions and eigenvalues of a stability problem and Chebyshev polynomial approximations to the same problem is their order of accuracy. Finite-difference approximations give only a finite order of accuracy in the sense that errors behave asymptotically like h^p for some finite p when the grid spacing h approaches zero. On the other hand, if the unperturbed velocity profile is smooth (infinitely differentiable), the Chebyshev polynomial approximations discussed here are of infinite order in the sense that errors decrease more rapidly than any power of $1/N$ as $N \to \infty$.

Another difference between finite-difference and Chebyshev polynomial approximations to stability problems concerns their resolution of possible regions of rapid change ('boundary layers') in the eigenfunctions. When the Reynolds number R (based on boundary layer thickness δ and freestream velocity U) is large, the eigenfunctions exhibit boundary layers of thickness of order $R^{-1/2}$ near $y = 0$ and internal layers of thickness of order $R^{-1/3}$ near the critical layer (where wave phase speed equals flow velocity).

In order for finite-difference approximations to be accurate, it is necessary that the grid spacing be at most $R^{-1/2}$ near y = 0 and at most $R^{-1/3}$ near the critical layer. Thus, if uniform grid spacing is used the number of grid points required is scaled by a factor $R^{1/2}$ as the Reynolds number increases. If non-uniform grid spacing is used, this difficulty may be partially avoided. On the other hand, the number of Chebyshev polynomials required for accurate stability calculations scales only as $R^{1/4}$ as the Reynolds number R approaches infinity.[10] In laminar flow control applications, this difference between finite-difference and Chebyshev polynomial methods is important. If the range of Reynolds numbers to be studied in a given LFC application is, say, R = 1000 —10,000 and if, say, 20 polynomials or 100 grid points are required to solve the problem at R =1000 (these resolutions are, in fact, typical), then less than 40 polynomials will be required at R=10,000 while more than 300 grid points will be required.

The rapid convergence properties of Chebyshev polynomials are verified as follows. If the unperturbed flow is smooth, then so are the eigenfunctions of the linearized Navier-Stokes equations. Let $T_n(x)$ denote the nth-degree Chebyshev polynomial of the first kind, defined by

$$T_n(\cos\theta) = \cos n\theta \qquad (5.1)$$

for all non-negative integers n. Some examples are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2-1$. It is possible to expand the eigenfunction $\psi(y)$ in the interval $-1 \le y \le 1$ (we discuss in

Sec. 6 techniques for handling the semi-infinite interval $0 \le y \le \infty$ encountered in boundary-layer stability problems) as

$$\psi(y) = \sum_{n=0}^{\infty} a_n T_n(y) \tag{5.2}$$

where

$$a_n = \frac{2}{\pi c_n} \int_{-1}^{1} \psi(y) T_n(y) (1-y^2)^{-1/2} \, dy \tag{5.3}$$

with $c_0 = 2$, $c_n = 1$ for $n > 0$. The rapidity of convergence of (5.2) for $|y| \le 1$ is easily demonstrated by observing that

$$f(\theta) = \psi(\cos \theta)$$

is an infinitely differentiable, even, periodic function of θ. Consequently, the theory of Fourier series ensures that $f(\theta)$ possesses a Fourier cosine expansion

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta \tag{5.4}$$

with the property that the error after N terms decreases more rapidly than any power of $1/N$ as $N \to \infty$. The expansion (5.4) is precisely (5.1) for $y = \cos \theta$.

The infinite-order accuracy of Chebyshev polynomial approximations to smooth functions holds no matter what the boundary values of the functions or their derivatives, in contrast to the situation when other classes of orthogonal functions (like trigonometric or Bessel functions) are used.[9-11]

In the following subsections, we discuss several programming and technical aspects of the application of Chebyshev polynomials to stability calculations.

Chebyshev Matrix Method

The derivation of the equations satisfied by the expansion coefficients a_n in the Chebyshev expansion

$$\psi(y) = \sum_{n=0}^{N} a_n T_n(y) \tag{5.5}$$

can be difficult and time-consuming if done by hand. In order to improve the flexibility of this method and the ease in which it can be applied to new problems, we have developed a nearly automatic matrix method for computer-generation of the equations satisfied by the coefficients a_n.

To illustrate the method, suppose we defined the vector $\hat{\psi}$ by

$$\hat{\psi} = \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ a_N \end{pmatrix} \tag{5.6}$$

When the expansion (5.5) is substituted into a differential operator, equations for the coefficients a_n are obtained by re-expanding the result in series of Chebyshev polynomials and equating coefficients of each Chebyshev polynomial: if

$$\mathcal{L}\psi = \sum_{n=0}^{\infty} b_n T_n(y)$$

then each b_n is a function of the a_n and we obtain N equations approximating $\mathcal{L}\psi = 0$ by setting $b_n = 0$ for $n \leq N$. In order to find these equations we must derive an efficient and easy method to determine the effect of the differential operator $\mathcal{L}$ on the vector $\hat{\psi}$.

Consider the Chebyshev representation of the function

$$\psi'(y) = \frac{d\psi(y)}{dy}$$

Let the associated vector of Chebyshev coefficients of $\psi'(y)$ be denoted by

$$\vec{\psi}' = \begin{pmatrix} a_0^{(1)} \\ a_1^{(1)} \\ \vdots \\ a_N^{(1)} \end{pmatrix}$$

Since

$$T_n' = 2n(T_{n-1} + T_{n-3} + \ldots),$$

it follows that

$$a_n^{(1)} = \frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+n \text{ odd}}} p a_p$$

where $c_0 = 2$, $c_n = 1$ for $n > 0$. Therefore,

$$\vec{\psi}' = D\vec{\psi}$$

where the N+1 x N+1 matrix D has elements

$$D_{ij} = \begin{cases} 0 & \text{if } i \geq j \text{ or } i+j \text{ is even} \\ \frac{2(j-1)}{c_{i-1}} & \text{otherwise} \end{cases}$$

Similarly, if $f(y) = y\psi(y)$, then the Chebyshev coefficients of f are given by

$$\vec{f} = Y\vec{\psi}$$

where the N+1 x N+1 matrix Y has elements

$$Y_{ij} = \begin{cases} 1 & \text{if } i=2,\ j=1 \\ 1/2 & \text{if } i=j+1 \text{ or } i=j-1 \\ 0 & \text{otherwise.} \end{cases}$$

The utility of setting up these matrices (and other similar ones) is the ease with which they may be used to set up the matrices expressing complicated differential operators. For example, the matrix for the operator

$yd^2/dy^2(y^3d/dy)$ is just YD^2Y^3D. (In fact, the last statement is not quite true because the matrix YD^2Y^3D does not correctly represent the action of the differential operator on high-order Chebyshev polynomials. However, it may be shown that this error is nearly negligible and we will not discuss it further here.)

In actual practice, it is frequently not necessary to store the matrices D, Y, etc., because their very simple form makes it possible to generate their elements as they are needed during the computation.

Let us consider how the Chebyshev matrix method applies to the solution of the Orr-Sommerfeld equation

$$(d^2/dy^2 - \alpha^2)^2\psi(y) = i\alpha R(\bar{u}(y)-c)(d^2/dy^2 - \alpha^2)\psi(y) - i\alpha R\bar{u}''(y)\psi(y),$$

where $\bar{u}(y)$ is the unperturbed profile. In terms of Chebyshev matrices, these equations are

$$(D^2 - \alpha^2 I)^2\vec{\psi} = i\alpha R(U - cI)(D^2 - \alpha^2 I)\vec{\psi} - i\alpha RU''\vec{\psi} \qquad (5.7)$$

where U is the matrix that multiplies by u(y) and U'' is the matrix that multiplies by U''.

Tau Method

The equations (5.7) for the Chebyshev coefficients a_n do not account for the boundary conditions imposed on $\psi(y)$. There are several ways to impose the boundary conditions consistently on (5.7); this can be done by Galerkin, collocation, or tau approximation.[11] It is usually most convenient to apply tau approximation, as we will now discuss.

The idea of the tau method is to drop enough of the equations (5.7) that all the boundary conditions can be applied. For the Orr-Sommerfeld equation, the boundary conditions

$$\psi(y) = \psi'(y) = 0$$

should be applied at the rigid boundaries $y = \pm 1$. Thus, we delete the last four rows of the matrix equation (5.7) and replace them by the four boundary conditions

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} (-1)^n a_n = 0$$

$$\sum_{n=0}^{N} n^2 a_n = \sum_{n=0}^{N} (-1)^{n+1} n^2 a_n = 0 \qquad (5.8)$$

Eqs. (5.8) follow because $T_n(\pm 1) = (\pm 1)^n$, $T_n'(\pm 1) = n^2(\pm 1)^{n+1}$. Thus, we retain N-3 equations of the form (5.7) and 4 equations of the form (5.8) so that there are N+1 equations for the N+1 unknowns $a_0, a_1, \ldots, a_N$.

Fast Fourier Transform

An additional advantage of Chebyshev polynomials over many other orthogonal bases is the existence of the fast Fourier transform to effect efficient conversion between the Chebyshev coordinates a_n and the physical space perturbation $\psi(y)$. In fact, since

$$\psi(\cos\theta) = \sum_{n=0}^{N} a_n \cos n\theta \ ,$$

Chebyshev series can be summed by any technique that sums Fourier cosine series, in particular, certain variants of the fast Fourier transform (see the Appendix of Ref. 12).

6. NUMERICAL METHODS FOR EIGENVALUE CALCULATIONS

We begin by making a number of general comments that apply to most numerical schemes for stability calculations. First, we remind the reader that the problem is difficult because the Orr-Sommerfeld equation (or other linearized equation) is moderately stiff at large Reynolds numbers, as reflected by the boundary layers exhibited by eigenfunctions (see Sec. 5) Second, calculations of moderately high accuracy are frequently needed for such purposes as computation of the group velocity (see Sec.7) and various optimization schemes (see Sec.7). Third, we comment that there are two possible kinds of stability calculations that can be made, spatial stability and temporal stability (see Sects. 2-3). For small growth rates, the results of temporal and spatial stability analyses are closely related.[13] However, because the differences between these two types of analyses may be central to the problem of LFC aircraft design, let us contrast them briefly here.

In a temporal stability analysis, the wavenumbers of the disturbance are assumed real while the frequency of the disturbance may be complex (a positive imaginary part indicates instability). On the other hand, in a spatial stability analysis, the frequency is taken real, while a positive imaginary part in a wavenumber indicates instability. Since many problems of aerodynamical interest are, on average, stationary in a suitable coordinate frame, it seems that

spatial stability analysis should be the more relevant one. However, this is by no means clear experimentally. It seems that low frequency disturbances are treated better by spatial theory than by temporal, but high frequency disturbances seem to agree better with temporal theory.[14]

The confusion between spatial and temporal theory is even more severe in the case of the propagation of three-dimensional disturbances, which are of primary interest in the boundary layers of LFC aircraft. Spatial stability theory is ambiguous for three-dimensional disturbances. In the case of two-dimensional disturbances, it is physically plausible that the direction of maximum growth of the disturbance is perpendicular to the constant-phase surfaces of the disturbance and parallel to the freestream flow direction. On the other hand, in three-dimensional layered flows, there is no apparent reason why the direction of maximum growth should be perpendicular to the direction of the constant-phase surfaces. If these directions are allowed to be arbitrary, one quickly gets involved in ill-posed mathematical problems. Until this basic question is resolved, it may be best to use temporal stability theory and a group velocity transformation for three-dimensional stability analyses of LFC boundary layers.

Finally, we comment on the mathematical technique to treat the semi-infinite domain $0 \leq y < \infty$ of the boundary layer. Grosch and Orszag[15] have shown that the best way to handle the y-direction is to transform it by means of the algebraic transformation

$$Y = 2\frac{y}{y+L} - 1 \qquad (6.1)$$

into the finite interval $-1 \leq Y < 1$, and then apply standard numerical techniques on the transformed finite interval. This technique provides accurate results with about 50% less resolution than required using simple truncation of the domain $0 \leq y \leq L$. Some results of a Chebyshev-spectral calculation are given in Table 1. Here we compare the accuracy of the most unstable mode of Blasius flow at a Reynolds number of 580 and a wavenumber of .179 using several treatments of the boundary at $y = \infty$. The methods are: (i) truncation, which involves solving the problem on the finite interval $0 \leq y \leq L$ for several values of L and with both no-slip $\psi(L) = \psi'(L) = 0$ boundary conditions and asymptotic boundary conditions $\psi'(L) + \alpha\psi(L) = 0$ applied; (ii) an exponential map of the form $Y = 1 - 2\exp(-y/L)$; and (iii)the algebraic transformation (6.1). For both methods (ii) and (iii), several kinds of boundary conditions are applied at $Y = 1$ ($y = \infty$), including no-slip conditions and <u>no</u> boundary conditions (!). We conclude from Table 1 that <u>no</u> boundary conditions at all need be applied at $Y = 1$.

With the mapping (6.1), L may be chosen to optimize the accuracy of the calculations. After some experience, it has been found that a good choice for L is $L \approx 2y_0$, where y_0 is the value of y at which the streamwise component of the velocity achieves 1/2 of its freestream value.

Table 4. Eigenvalues of the Orr-Sommerfeld equation for Blasius flow, R = 580, α = 0.179.

Case	Mapping	Boundary Conditions at 'z = ∞'	L	N (Number of Chebyshev Modes)	c
1	Truncation	$\phi(L) = \phi'(L) = 0$	10	44	0.37887 7 + i0.00025 0
2		"	20	44	0.36455 7 + i0.00777 3
3		"	20	46	0.36455 1 + i0.00778 1
4		"	30	44	0.36399 6 + i0.00788 8
5		$\phi'(L) + \alpha\phi(L) = 0$	20	44	0.36021 3 + i0.00667 1
6		"	30	44	0.36404 1 + i0.00811 3
7	Exponential	$\phi(1) = 0(1)$	1	42	0.34858 0 + i0.01312 9
8		"	1	46	0.34961 1 + i0.01285 6
9		$\phi(1) = \phi'(1) = 0$	1	46	0.38378 9 - i0.00276 6
10		"	1	70	0.37853 1 + i0.00047 1
11	Algebraic	$\phi(1) = 0(1)$	1	26	0.36414 7 + i0.00800 7
12		"	1	34	0.36412 1 + i0.00795 76
13		"	1	42	0.36412 288 + i0.00795 975
14		$\phi(1) = \phi'(1) = 0$	1	42	0.36412 325 + i0.00795 894
15		"	1	60	0.36412 285 + i0.00795 973
16		$\phi(1) = 0$	1	42	0.36412 287 + i0.00795 976

Global Methods for Temporal Eigenvalue Calculations

When no guess is available for the eigenvalue of interest, it is best to use a method that is globally convergent and nearly guaranteed to converge to the eigenvalue. Such a method may be based on the matrix QR algorithm[16] for calculation of the eigenvalues of a general complex matrix.

When the Orr-Sommerfeld equation (or other similar differential equation) is formulated as a matrix problem (using either Chebyshev polynomials or finite-difference methods), it takes the form

$$Ax = \lambda Bx \tag{6.2}$$

where λ is the eigenvalue (denoted c or ω above in the case of temporal stability calculations) and x is the discrete representation of the eigenfunction. The eigenvalue is determined by the determinant condition

$$\text{Det } |A - \lambda B| = 0. \tag{6.3}$$

Eq. (6.3) is a generalized eigenvalue problem and the matrix QR algorithm does not apply directly unless either A or B is invertible. (If, say, B^{-1} exists then (6.3) is equivalent to the standard eigenvalue problem

$$\text{Det } |B^{-1}A - I| = 0.)$$

However, it is frequently the case that A and B are singular and a more general method must be developed.

To solve the generalized eigenvalue problem with singular A and B, we proceed as follows. There are two steps that are

executed recursively.

(i) We use fully pivoted row operations to reduce B to upper triangular form, executing the same row operations on the matrix A. The resulting generalized eigenvalue problem Det $|A' - \lambda B'| = 0$ has the same eigenvalues as (6.3) because the same row operations were performed on A and B. If all the diagonal elements of B' are nonzero then B' (and hence B) is nonsingular and the problem can be immediately reduced to the standard eigenvalue problem (and then solved by the QR algorithm). Thus, let us assume that all elements b'_{ij} with $K \leq i \leq N$ are zero (if any elements of this matrix were not zero, full pivoting would ensure additional nonzero diagonal elements).

(ii) We perform fully pivoted column operations on the rows $j=K,\ldots,N$ of the matrix A' to transform A' into an upper triangular matrix A". The same column operations are performed on B" but it is still true that $b''_{ij} = 0$ for $K \leq i \leq N$ and all j. The generalized eigenvalue problem Det $|A'' - \lambda B''| = 0$ has the same eigenvalues as (6.3) because it is obtained from it by row and column operations simultaneously on both A and B. However, rows K,..., N of A" - λB" are upper triangular, so the generalized eigenvalue problem for A" and B" has a solution only when the generalized eigenvalue problem

$$\text{Det } |A''_K - \lambda B''_K| = 0$$

has a solution, where A''_K and B''_K are the K-1 x K-1 dimensional matrices obtained from A" and B" by discarding rows and columns K, K+1,..., N.

(iii) Go to step (i). Eventually B' must be non-singular or there are no generalized eigenvalues or all λ are generalized eigenvalues.

Some results of using the above algorithm on a CDC 7600 computer are given in Table 2.†

Table 2.

Timings of the Global Eigenvalue Program

	26 x 26 matrix	43 x 43 matrix
QR method (Fortran, all eigenvalues)	0.15 s	0.55 s
Generalized eigenvalue problem (invertible A or B)	0.18s	0.66 s
Generalized eigenvalue problem (singular A and B) Two recursions	0.27s	1.1s

The advantage of this technique is that it is very general and is globally convergent. The disadvantage is that it is not too fast (though it is probably faster when combined with Chebyshev polynomials than most commonly used methods).

When finite-difference methods are used, it may be better to obtain a globally convergent result by use of the LR algorithm and band matrix factorizations. We have not yet tried this technique and cannot yet comment on it.

†A much simpler technique to solve the generalized eigenvalue problem has been suggested by J. Shearer (private communication, 1977). If μ is not one of the eigenvalues of (6.2) then $A - \mu B$ is invertible. If the eigenvalues of $(A-\mu B)^{-1}B$ are denoted by c, then the eigenvalues of (6.2) are $\lambda = \mu + 1/c$.

Spurious Unstable Modes

One of the drawbacks of the global method as formulated above is that the generalized eigenvalue problem (6.3) may indicate the existence of growing (unstable) modes that are not physically relevant. These spurious unstable modes, which may appear for either the Chebyshev-spectral or finite-difference methods, do not correspond to solutions of the differential equation--as the spatial resolution used to discretize the eigenfunction changes (i.e. the number of grid points or Chebyshev polynomials), true modes of the differential equation converge while spurious modes do not.

A clumsy way to distinguish spurious modes from true modes is to change the spatial resolution and retain only those modes that do not change appreciably. This is neither efficient nor elegant.

A better way is to eliminate the spurious unstable modes entirely. Spurious stable modes are still possible, but since these stable modes are normally very stable, they are not of much interest and can be easily disregarded without testing their true nature. We shall now describe a technique for eliminating the spurious unstable modes.

The idea is simply that the spurious unstable modes would, if we used the same numerical method used for the stability problem on an initial-value problem instead, cause the unconditional instability of the numerical solution of the initial-value problem. On the other hand, if we were careful enough to use a numerical method for the stability problem that was also numerically stable for the initial-value

problem, then no spurious unstable modes would exist.

There are several ways to eliminate the spurious unstable modes in the Chebyshev spectral methods outlined in Sec. 5. One way is to use a Galerkin procedure instead of the tau procedure discussed in Sec. 5.[11] Another way is to factor the fourth-order Orr-Sommerfeld equation into two second-order equations and then apply the tau method to each of the second-order equations. Thus, the usual procedure is to apply four boundary conditions to the fourth-order Orr-Sommerfeld equation

$$(\nabla^2\psi)_t + \bar{\psi}_y\nabla^2\psi_x - \psi_x\nabla^2\bar{\psi}_y = \nu\nabla^4\psi$$

This procedure usually leads to spurious unstable modes. However, rewriting the Orr-Sommerfeld equation as the two second-order equations

$$\zeta = \nabla^2\psi$$

$$\zeta_t + \bar{\psi}_y\zeta_x - \psi_x\bar{\zeta}_y = \nu\nabla^2\zeta$$

and then applying the boundary conditions $\psi = 0$ to the first equation (because this first equation is equivalent to the incompressibility constraint and the associated boundary condition should be zero normal flow) and $\psi' = 0$ to the second equation (because the second equation embodies the viscous, frictional effects and the associated boundary condition is no-slip) gives no spurious unstable modes.

Local Methods for Temporal Eigenvalue Calculations

There are many local methods (in which a reasonable guess is available) for eigenvalue problems. Among the well-known methods that have been implemented for difference methods are the methods of orthogonalization[17] and parallel shooting[18] together with either a bisection search or a Newton's method search. There are several effective computer codes that implement these procedures, including the SUPORT code (written by Scott and Watts of Sandia Laboratory), the TAPS code (written by Gentry and Wazzan of McDonnell Douglas Corp.) and codes by Mack of Jet Propulsion Laboratory and Keller and Cebeci. At the end of this section, we will cite some experience we have had with the SUPORT code and quote some private communications concerning the efficiency of the other codes.

Another way to perform a local analysis is to use a simple iterative method to find the eigenvalues of the matrix equation (6.3) that approximates the Orr-Sommerfeld equation. An effective and efficient procedure for doing this is to use the inverse iteration procedure[16]:

$$(A - \lambda_k I)x_{k+1} = cx_k \tag{6.4}$$

$$(A - \lambda_k I)^T y_{k+1} = c'y_k \tag{6.5}$$

$$\lambda_{k+1} = y_{k+1}^T Ax_{k+1}/y_{k+1}^T x_{k+1} \tag{6.6}$$

The procedure (6.4-6) is very effective once a good guess for an eigenvalue is available because the convergence is cubic:

$$\lambda_{k+1} - \lambda = 0((\lambda_k - \lambda)^3) .$$

Here c and c' are normalization constants so x_k and y_k are normalized.

The method (6.4-6) is designed to work efficiently even with non-symmetric matrices A. Eq. (6.4) shoudl be solved using a fully pivoted LU method, in which case the same LU factorization applies to the solution of (6.5).

In practice, it is not necessary to update the eigenvalue approximation λ_k after each iteration. In fact, we have found it to be most efficient to iterate (6.4-5) approximately 5-10 times while keeping λ_k fixed (and, therefore, using the same LU factorization of $A - \lambda_k I$).

The generalization of (6.4-6) to the generalized eigenvalue problem (6.3) is:

$$(A - \lambda_k B)x_{k+1} = cBx_k \tag{6.7}$$

$$(A - \lambda_k B)^T y_{k+1} = c'B^T y_k \tag{6.8}$$

$$\lambda_{k+1} = y_{k+1}^T Ax_k / y_k^T Bx_k \tag{6.9}$$

The algorithm (6.7-9) still has a rapid rate of convergence because it is equivalent to Newton's method for the solution of the generalized eigenvalue problem (6.3) (see below). Some data on the speed of the algorithm (6.7-9) for Chebyshev methods applied to the stability of interior flows is given in Table 3.

Table 3. Timings of the Local Eigenvalue Program

	26 x 26 matrix	43 x 43 matrix
Fortran program 1 eigenvalue good guess available final accuracy 10^{-8}	0.03 s	0.13 s
Assembly LU program otherwise same	0.02 s	0.08 s

Spatial Stability Calculations

The Orr-Sommerfeld equation for spatial stability calculations involves a nonlinear, quartic polynomial eigenvalue problem of the form

$$A(\lambda)x \equiv (\lambda^4 A_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda^1 A_1 + A_0)x = 0 \qquad (6.10)$$

Global methods for the solution of nonlinear eigenvalue problems like (6.10) may be inefficient. A simple global method is to set

$$x_1 = x$$

$$x_2 = \lambda x_1 \qquad (6.11)$$

$$x_3 = \lambda x_2 \qquad (6.12)$$

$$x_4 = \lambda x_3 \qquad (6.13)$$

$$A_3 x_4 + A_2 x_3 + A_1 x_2 + A_0 x_1 = -\lambda A_4 x_4 \qquad (6.14)$$

and then formulate (6.11-14) as a generalized eigenvalue problem of the form (6.3) involving 4N x 4N matrices:

$$\begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ A_0 & A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \lambda \begin{pmatrix} I & C & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -A_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The obvious disadvantage of this method is that it requires 16 times as much memory as (6.3) requires and 64 times as much computer times. It is not practical except for problems of extreme urgency, in which case it has the virtue of easy programming (so long as N is not too big).

A local method may be based on Newton's method for the solution of the nonlinear coupled system of equations

$$A(\lambda)\, x = 0 \tag{6.15}$$

$$A(\lambda)^T y = 0 \tag{6.16}$$

$$y^T x = 1 \tag{6.17}$$

so that y is the adjoint eigenvector to x. Eqs. (6.15-17) represent 2N+1 equations for the 2N+1 unknowns $x_1,\ldots,x_N,y_1,\ldots,y_N,\lambda$. Newton's method involves linearizing (6.15)-(6.17) about an approximate solution $x^{(0)},y^{(0)}, \lambda^{(0)}$:

$$A(\lambda)x = A(\lambda^{(0)})x^{(0)} + A(\lambda^{(0)})(x-x^{(0)}) + A'(\lambda^{(0)})x^{(0)}(\lambda-\lambda^{(0)}) + \ldots = 0 \tag{6.15'}$$

$$y^T x = y^{(0)T}x^{(0)} + y^{(0)T}(x-x^{(0)}) + (y-y^{(0)})^T x^{(0)} + \ldots = 1 \tag{6.17'}$$

Thus, if we insist that the new approximation $x^{(1)}, y^{(1)}, \lambda^{(1)}$ be such that the linear approximations (6.15'-17') are satisfied then we obtain the following iteration scheme:

$$A(\lambda_k)x_{k+1} = cA'(\lambda_k)x_k \tag{6.18}$$

$$A(\lambda_k)^T y_{k+1} = c'A'(\lambda_k)^T y_k \tag{6.19}$$

$$\lambda_{k+1} = \lambda_k - y_{k+1}^T A(\lambda_k)x_{k+1}/y_{k+1}^T A'(\lambda_k)x_k \tag{6.20}$$

The advantages of this method are that it is essentially as fast as the local method for the linear generalized eigenvalue problem (6.7-9) and that it has low memory requirements. The disadvantage is that the initial guess sometimes has to be quite good for it to work.

Our recommendation for the most efficient technique for spatial stability analysis is as follows:

(i) If no good approximation to the eigenvalue is available, perform a temporal stability analysis using a globally convergent algorithm. Then calculate the group velocity using the methods to be discussed in Sec. 7 and transform this temporal mode into an approximate spatial mode using Gaster's group velocity transformation[13].

(ii) If a good guess is available (say by method (i)) then use the local algorithm (6.18-20) to improve the eigenvalue approximation.

Comparison of Numerical Methods for Plane Poiseuille Flow

In this subsection, we present some numerical results concerning the stability of plane Poiseuille flow. In Table 4, we list the Chebyshev approximation to the most unstable mode of plane Poiseuille flow at R = 10,000 with wavenumber $\alpha = 1$ as a function of the number of retained Chebyshev polynomials (here we have assumed that the mode being sought is symmetric in y so we actually retain Chebyshev polynomials up to degree 2M). It is apparent that accurate results are achieved rapidly as the number of retained polynomials increases. Similar results have been obtained at Reynolds numbers of 10^6 and higher; at $R = 10^6$, 50 Chebyshev polynomials yield an eigenvalue accurate to about 1 part in 10^4 (this calculation using a global code requires less than 1.5 s of CDC 7600 time).

In Table 5, we list some results and computer times obtained using the SUPORT code mentioned above together with a Newton's method iteration procedure for the eigenvalue. Although we do not have the data, similar results and computer timings have been reported by other groups using finite-

Table 4 . Chebyshev approximation to the most unstable mode of plane Poiseuille flow for $\alpha = 1$, $R = 10000$.

M + 1	λ
17	.23713751 + i.00563644
20	.23752676 + i.00373427
23	.23752670 + i.00373982
26	.23752648 + i.00373967
29	.23752649 + i.00373967
38	.23752649 + i.00373967
50	.23752649 + i.00373967

difference codes. Mack reports that it requires about 3 s of UNIVAC 1108 time to converge to an accurate eigenvalue given a reasonably good guess; Keller and Cebeci achieve accurate results at Reynolds numbers of order 10^4 in about 0.9 s of CDC 6600 time. Since 1 s of CDC 7600 time equals about 5 s of CDC 6600 time and about 12 s of 1108 time, we see that the local finite-difference methods are comparable in speed to the global Chebyshev methods. However, it does seem that the local Chebyshev methods discussed above are faster by about an order of magnitude. But it should be pointed out that these comparisons are perhaps being done unfairly: M. Scott who wrote SUPORT informs us that SUPORT can be speeded up by about a factor 2 by simply changing the time-stepping algorithm; the Chebyshev codes can also be speeded up considerably in the areas of matrix set-up and full matrix pivoting.

Table 5. Some Results of the SUPORT Code

Reynolds number	Error	#of grid points	#of iterations	CDC7600 time/eigenvalue
10^4	10^{-4}	140	4	1.4 s
10^6	10^{-4}	1400	6	23 s

7. NUMERICAL METHODS: EIGENVECTORS, GROUP VELOCITY, AND OPTIMIZATION METHODS

Eigenvectors

There are several efficient numerical methods to compute eigenvectors. In any of the standard finite-difference methods, the eigenvector $\phi(y)$ is computed as part of the calculation of the eigenvalue λ .. Also, in the local matrix iterative methods advocated in Sec.6, the eigenvector is found as part of the iteration determining the eigenvalue λ .

A separate calculation of the eigenvector is required only for the case in which global matrix eigenvalue methods are used to determine λ . In this case, one or two applications of the inverse iteration

$$(A - \lambda B)x^{(k+1)} = cBx^{(k)} \tag{7.1}$$

where $x^{(0)}$ is arbitrary usually determine the eigenvector to within roundoff error. The value of λ used in (7.1) may be the calculated eigenvalue to machine precision--this normally does not cause any problems with singular matrices. For a 43 x 43 matrix, the calculation of the eigenvector by this inverse iteration scheme requires about 75 ms using an assembly language fully pivoted LU algorithm on a CDC 7600.

The computation of the eigenvector of the adjoint Orr-Sommerfeld equation may be done similarly (noting that the spectrum of the adjoint is the same as the spectrum of the Orr-Sommerfeld equation up to complex conjugation).

In practice, calculation of the adjoint egenfunction using the Chebyshev-spectral method requires about 150 ms using the assembly LU program because the matrix of the adjoint operator must be set up.

Group Velocity

The group velocity is of importance in relating the results of spatial and temporal stability theory and in several optimization problems (see below). In a layered flow with three-dimensional disturbances having wavevector (α, β) and frequency $\omega(\alpha, \beta)$, the group velocity $\vec{v}_g$ is

$$\vec{v}_g = \left(\frac{\partial \omega}{\partial \alpha}, \frac{\partial \omega}{\partial \beta}\right) \tag{7.2}$$

One way to compute the group velocity is simply to compute the frequency ω for several nearby values of α, β and then use finite-difference approximations to $\vec{v}_g$. This procedure is neither efficient nor elegant.

A much better way to compute the group velocity will now be described. It is usually faster than the crude finite-difference method described above (except possibly for the fastest local iterative eigenvalue solvers). We start by writing the Orr-Sommerfeld equation for three dimensional disturbances in the form

$$\mathcal{L}\psi \equiv \{(D^2 - \alpha^2 - \beta^2)^2 - iR[(\alpha\bar{u} + \beta\bar{w} - \omega)(D^2 - \alpha^2 - \beta^2) - (\alpha\bar{u}'' + \beta\bar{w}'')]\}\psi(y) = 0 \tag{7.3}$$

Taking the derivative of (7.3) with respect to α, we obtain

$$\mathcal{L}\left(\frac{\partial \psi}{\partial \alpha}\right) = \{4\alpha(D^2 - \alpha^2 - \beta^2) + iR(\bar{u} - \frac{\partial \omega}{\partial \alpha})(D^2 - \alpha^2 - \beta^2) - \bar{u}'' - 2\alpha(\alpha\bar{u} + \beta\bar{w} - \omega)\}\psi(y) \tag{7.4}$$

Therefore, if χ is the adjoint eigenfunction to ϕ, we obtain

$$\frac{\partial \omega}{\partial \alpha} = -\frac{4i\alpha}{R} + \frac{\int \chi\{\bar{u}(D^2 - \alpha^2 - \beta^2) - \bar{u}'' - 2\alpha(\alpha\bar{u} + \beta\bar{w} - \omega)\}\phi \, dy}{\int \chi(D^2 - \alpha^2 - \beta^2)\phi \, dy} \tag{7.5}$$

There is a similar expression for $\partial\omega/\partial\beta$.

The computation of the group velocity using (7.5) requires little additional computational work to the calculation of the adjoint eigenfunction.

Optimization Methods

There are several kinds of information concerning the stability of a given flow that may be of interest. Some examples are:

(i) A plot of α vs β at fixed R and frequency Re ω.

(ii) Determination of that α and β that maximize Im ω at given R and Re ω.

(iii) For given Im ω a plot of α vs β at fixed R or $\alpha^2 + \beta^2$ vs R at fixed disturbance propagation angle or a neutral stability curve (α vs R for a two-dimensional disturbance with Im $\omega = 0$).

(iv) Determination of the critical Reynolds number, i.e. the smallest value of R at which there is a mode with Im $\omega = 0$.

(v) Computation of nonlinear and non-parallel flow terms.

To illustrate how these problems can be solved efficiently on a computer, let us consider the solution of the problem (i).

An efficient procedure for obtaining the curve (i) of those α and β having Re ω = f at a fixed Reynolds number R is to use Newton's method as follows:

(a) Starting with an approximation α_0, β_0 to a point on the curve Re ω = f, we compute the group velocity at α_0, β_0 and obtain the new approximation

$$(\alpha_1,\beta_1) = (\alpha_0,\beta_0) - \frac{f - \text{Re}\,\omega_0}{|\text{Re}\,\vec{v}_g|^2}\,\text{Re}\,\vec{v}_g \qquad (7.6)$$

by shooting along the normal to the curve Re ω = f.

(b) Repeat step (a) until the approximation lies within a given error tolerance of the desired curve Re ω = f.

(c) Obtain an approximation to a new point on the curve Re ω = f by shooting along the tangent to the curve from the previously found point; the tangent to the curve is in the direction $\text{Re}(\frac{\partial\omega}{\partial\beta}, -\frac{\partial\omega}{\partial\alpha})$. Repeat steps (a) and (b).

The solution of problem (ii) is obtained similarly. Here it is possible to choose a new point on the curve Re ω = f with a more favorable Im ω by use of either Newton's method (which requires second derivatives of ω) or, say, cubic interpolation. One word of caution on this procedure for the solution of problem (ii)is that very small errors in the value of Re ω may confuse the search for the maximum of Im ω . The reason is simply that in typical stability problems, Re ω >> Im ω . The moral is: the determination of optimal properties of the stability characteristics of flows requires extremely high accuracy calculations.

8. SUMMARY OF RESULTS

A summary of our conclusions for a moderately fast stability code running on a CDC7600 computer is given in Table 6 for incompressible flow and Table 7 for compressible flow.

Table 6. Typical computer timings for incompressible Blasius flow at Reynolds number of order 10^3 with numerical errors of order 10^{-6} (Chebyshev polynomial methods with N = 34 polynomials)

Operation	Computer Time
Temporal eigenvalue	
Matrix set-up	0.09 s
Global search	0.60 s
Local search	0.04 s
Eigenvector	
Global search	0.03 s
Local search	--
Adjoint eigenvector	0.09 s
Group velocity	0.10 s
Nonlinear stability terms	0.12 s
Nonparallel flow terms	0.04 s
Spatial eigenvalue	
Global search	40 s
Local search (first guess by temporal code)	0.4 s
Local search (guess known)	0.04 s

Table 7. Typical computer timings for compressible flow stability calculations at Reynolds numbers of order 10^3 with errors of order 10^{-6} (Chebyshev polynomial methods with N = 34 polynomials)

Operation	Timing for Two-Dimensional Modes	Expected Timing for Three-Dimensional Modes
Temporal eigenvalue		
Global search	30 s	60 s
Local search		
Direct	2 s	8 s
Indirect	0.25 s	0.3 s
Eigenvector and adjoint	0.3 s	0.35 s
Nonlinear and non-parallel terms by transform method	0.4 s	0.5 s

Final Remarks

(A) Roundoff Error: The effects of roundoff error become more acute as the matrix size increases. This effect is shown in Table 8. We conclude that high machine accuracy is necessary to even attempt calculations at large Reynolds numbers.

(B) Accuracy of Profiles: We have computed errors in the imaginary part of eigenvalues of as large as 10% with errors in the imposed profiles as small as .01%. However, these dangerous perturbations are of a very special kind wherein they are concentrated near the critical layer of the mode. (Obviously, if we perturb $\bar{u}(y)$ by an arbitrarily small amount, we may perturb $\bar{u}''(y)$ by an arbitrarily large

Table 8 . Effect of roundoff error on the most unstable mode of plane Poiseuille flow for $\alpha = 1$, $R = 10000$.

M + 1	λ(roundoff = 10^{-8})	λ(roundoff = 10^{-12})
20	.23752685 + i.00373451	.23752676 + i.00373427
23	.23754139 + i.00383489	.23752670 + i.00373982
26	.23749300 + i.00368897	.23752646 + i.00373965
38	.23714159 + i.00352930	.23752648 + i.00373966
44	.23348160 + i.00534311	.23752648 + i.00373965
50	.23813295 - i.00296263	.23752655 + i.00373979

amount at the same time. However, the dangerous perturbations cited above are perturbations in u", but very noisy perturbations in u" near the critical layer.)

In order to avoid these difficulties, it is necessary that the mean velocity profiles used in the stability calculations be very smooth and that the second derivatives of these profiles also be very smooth.

(C) Accuracy of Eigenvalues: Because the real parts of eigenvalues are typically much larger than the imaginary parts of the eigenvalues, it is necessary to maintain very high accuracy in stability calculations in order to get any meaningful result concerning instability.

(D) Nonlinear and Non-Parallel Flow Terms: The inclusion of these terms requires little additional work to that already expended in computing the eigenvalues of the flow. Therefore, it seems to make good sense to compute these terms.

(E) Typical Computer Times: It is realistic to expect a well-conceived stability code for incompressible flows to require less than 1/2 s per station of CDC 7600 time; for three-dimensional modes of compressible flows, about 5 s per station is realistic. These estimates are for total computer time per station across a typical LFC wing provided either that an approximate eigenvalue is available at the leading edge or a previous station has been calculated nearby so that local methods may be used.

REFERENCES

1. J.T. Stuart, "On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part I. The basic behavior in plane Poiseuille flow." J. Fluid Mech. 9,353(1960).

2. J. Watson, "On the nonlinear mechanics of wave disturbances in stable and unstable parallel flows. Part II. The development of a solution for plane Poiseuille flow and for plane Couette flow." J. Fluid Mech. 9,371(1960).

3. D.J. Benney, "A nonlinear theory for oscillations in a parallel flow." J. Fluid Mech. 10,209(1961).

4. J.T. Stuart, " On finite amplitude oscillations in laminar mixing layers." J. Fluid Mech. 29,417(1967).

5. D.J. Benney and R.F. Bergeron, Jr., "A new class of nonlinear waves in parallel flows." Stud. in Appl. Math. 48, 181(1969).

6. D.J. Benney and S.A. Maslowe, "The evolution in space and time of nonlinear waves in parallel shear flows." Stud. in Appl. Math. 54, 181 (1975).

7. W.S. Saric and A.H. Nayfeh, " Nonparallel stability of boundary-layer flows." Phys. Fluids 18, 945(1975).

8. J.T. Stuart, "Nonlinear stability theory." Ann. Rev. Fluid Mech. 3, 347(1971).

9. S.A. Orszag, "Accurate solution of the Orr-Sommerfeld stability equation." J. Fluid Mech. 50, 689(1971).

10. S.A. Orszag and M. Israeli, "Numerical simulation of viscous incompressible flows." Ann. Rev. Fluid Mech. 6, 281(1974).

11. D. Gottlieb and S.A. Orszag, Numerical Analysis of Spectral Methods, SIAM-CBMS, c.1977.

12. S.A. Orszag, "Numerical simulation of incompressible flows within simple boundaries. I. Galerkin (spectral) representations." Stud. in Appl. Math. 50, 293(1971).

13. M. Gaster,"A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability." J. Fluid Mech. 14, 222(1962).

14. A. Michalke,"On spatially growing disturbances in an inviscid shear layer." J. Fluid Mech. 23, 521(1965).

15. C.E. Grosch and S.A. Orszag, "Numerical solution of problems in unbounded regions: coordinate transforms." J. Comp. Phys., Oct., 1977.

16. J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford (1965).

17. R. Betchov and W.O. Criminale, Jr., Stability of Parallel Flows, Academic (1967).

18. H.B. Keller, Numerical Methods for Two-Point Boundary Value Problems, Blaisdell (1968).

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